Did Cantor Sow the Seed of Measure and Integral?

M. G. Nadkarni
Department of Mathematics
University of Mumbai
Kalina, Mumbai, India 400 098
nadkarni@math.mu.ac.in

1. Introduction, Derived Sets

The purpose of this article is to discuss Cantor's notion of volume of an arbitrary bounded subset of $\mathbb{R}^n$ and relate it to the subsequent discovery and development of measure theory and integration.

Cantor's major preoccupation during an important period of his creativity was the system of derived sets of a set in $\mathbb{R}^n$, and perfect sets which arise as the ultimate derived set if the set is not countable. In the course of his study of derived sets Cantor discovered well ordered sets other than positive integers, in particular the first uncountable ordinal, whose subsets are needed to properly index derived sets. The derived set of a bounded set P in $\mathbb{R}^n$, denoted by P' or P^1, is by definition the set of limit points of P; P' is always closed, irrespective of whether P is closed or not, and the derived set of P is same as the derived set of the closure of P. So one may assume, if one wishes, that P is already closed, although Cantor does not make this assumption and we will not either.

Further, P'' or P^2 defines the derived set of P' and, in general if we have defined P^α for all countable ordinals $\alpha < \beta$, then the set

$$P^\beta = \Big(\bigcap_{\alpha<\beta} P^\alpha\Big)'$$

defines the derived set of P corresponding to the countable ordinal β. It is easy to see that $P^\beta \subset P^\alpha$ if $\alpha < \beta$, so that if P' is countable then there is a smallest countable ordinal γ such that $P^\gamma = P^{\gamma+1}$.

In case P' is not countable, we can speak of the set C of condensation points of P', namely those points $x \in P'$ every neighbourhood of which contains uncountably many points of P'. We note that the set $B = P' - C$ is countable, and that C is perfect in the sense that C is closed and every point of C is a limit point (in fact a condensation point) of C. Now

$$P' = B \cup C,$$
$$P'' = B' \cup C' = B' \cup C,$$

and more generally for any countable ordinal β,

$$P^\beta = B^\beta \cup C.$$

Since B is countable, and a condensation point of B^α must necessarily be in C for any $\alpha \geq 1$, we see again that there is a smallest countable ordinal γ such that $B^\gamma = B^{\gamma+1}$ so that

$$P^\gamma = P^{\gamma+1},$$

and for all $\delta > \gamma$, $P^\delta = P^\gamma$. Having established these facts, not necessarily in exactly this manner, Cantor takes up the question of the size of P and P', in particular the cardinality of P' and the metrical size of P and P', and it is these considerations of Cantor which are the focus of this article.

2. Cantor's Answer to an Under-Graduate Exam. Question in Riemann Integration Theory

The first theorem in this direction is the following, which points towards Heine–Borel theorem for a neater proof, but it is good that such a proof was not available, for Cantor's proof also suggests a question concerning derivatives which Lebesgue answered so very well in its fullness and depth. Let us note that the theorem below, or weaker versions of it, are favourite questions in the under-graduate exams on Riemann integration theory.

Theorem ([1], p. 161, 1883). *If a linear point set $P \subset (a, b)$ is such that P' is countable, then it is always possible to put P inside a finite union of intervals with sum of interval lengths arbitrarily small.*

Proof. Assume without loss of generality that $(a, b) = (0, 1)$. Since P' is countable, so is P, and hence also $Q = P \cup P'$ and we have, with $R = (0, 1) - P$,

$$(0, 1) = Q \cup R.$$

We enumerate Q as $u_1, u_2, u_3 \dots$. Since Q is closed, R is open, and therefore it is a union of a finite or countable number of pairwise disjoint open intervals (c_ν, d_ν), $\nu = 1, 2, 3, \dots$, where c_ν, d_ν belong to Q for all ν. But not every point in Q is necessarily a c_ν or a d_ν since a point of Q may be a limit point of P. The length of the interval (c_ν, d_ν) is

$$d_\nu - c_\nu.$$

The sum of lengths of these intervals we call σ so that

$$\sum_{\nu=1}^{\infty}(d_\nu - c_\nu) = \sigma.$$

Clearly $\sigma \leq 1$. We will show that $\sigma = 1$ so that the possibility $\sigma < 1$ is eliminated. To this end define a function f on $0 \leq x \leq 1$ as follows: $f(0) = 0$, and for $0 < x \leq 1$, $f(x) =$ sum of the lengths of intervals (c_ν, d_ν) which lie in the interval $(0, x)$, where if x is in the interval (c_λ, d_λ) for some λ then we add only the part (c_λ, x) of (c_λ, d_λ) to the sum, (i.e., we add only $(x - c_\lambda)$ to the sum). We have $f(0) = 0$, $f(1) = \sigma$ and f is continuous on $0 \leq x \leq 1$. From the definition of f it follows easily that if $x, x + h$ are in $(0, 1)$, $h > 0$, then

$$f(x + h) - f(x) \leq h,$$

from which the continuity of f follows. Further if $x, x + h$ are in the same interval (c_ν, d_ν), then

$$f(x + h) - f(x) = h,$$

and

$$(x + h) - f(x + h) = x - f(x).$$

Consider the function

$$\phi(x) = x - f(x).$$

It is continuous on $0 \leq x \leq 1$ and so assumes every value between $\phi(0) = 0$ and $\phi(1) = 1 - \sigma$. Also ϕ is constant on the intervals (c_ν, d_ν), so that on the set $\cup_{\nu=1}^{\infty}(c_\nu, d_\nu) = (0, 1) - Q$, ϕ assumes only a countable number of values. These values, together with the values assumed by ϕ on the countable set Q, form the set of all values assumed by ϕ, so the set of values assumed by ϕ is countable. On the other hand if $\sigma < 1$, the set of values assumed by ϕ is the uncountable set of all real numbers from 0 to $1 - \sigma$. The contradiction shows that $\sigma = 1$, and this is enough according to Cantor to conclude the theorem.

A short proof is supplied by E. Zermelo, editor of ([1], p. 164). Since $\sum_{\nu=1}^{\infty}(d_\nu - c_\nu) = 1$, given $\epsilon > 0$ choose N so large that $\sum_{\nu=1}^{N}(d_\nu - c_\nu) > 1 - \frac{1}{2}\epsilon$. Then

$$P \cup P' \subset E =_{def} (0, 1) - \bigcup_{\nu=1}^{N}(c_\nu, d_\nu),$$

and E is made of finitely many pairwise disjoint intervals with sum of their lengths $\leq \frac{1}{2}\epsilon$. We can enlarge these intervals whose union is E slightly so that each of them is open and sum of their lengths remains less than ϵ. Thus the theorem is proved. □

It is important not to forget this proof for one year later in 1884 ([1] p. 259) Cantor says that the same proof will show the following:

Let $P \subset (0, 1)$ be any set, $P \cup P'$ its closure and let (c_ν, d_ν), $\nu = 1, 2, 3, \dots$ be the pairwise disjoint intervals whose union is $(0, 1) - P \cup P'$. Let σ denote the sum $\sum_{\nu=1}^{\infty}(d_\nu - c_\nu)$. Then given any $\epsilon > 0$ we can find finitely many intervals which cover $P \cup P'$ and such that sum of their interval lengths is less than $1 - \sigma + \epsilon$. We will use this fact in a while.

The functions f and ϕ can be defined in this situation too and will satisfy for $h > 0$ for which $x, x + h$ are both in $(0, 1)$ the inequalities

$$f(x + h) - f(x) \leq h,$$

$$\phi(x + h) - \phi(x) \leq h$$

which at once suggests the question whether

$$\lim_{x \to 0} \frac{\phi(x + h) - \phi(x)}{h}$$

exists, answer to which is given by the celebrated Lebesgue density theorem, which is a mathematical way of expressing the statement "line has length but no breadth."

We also note that although Cantor calls the numbers $(d_\nu - c_\nu)$ the size (Groesse) of the interval (c_ν, d_ν), and he writes down the $\sum_{\nu=1}^{\infty}(d_\nu - c_\nu)$ of sizes (intervalgroessen) of the pairwise disjoint intervals (c_ν, d_ν), $\nu = 1, 2, 3, \dots$, he does not use this sum to define the size or groesse of the open set $\cup_{\nu=1}^{\infty}(c_\nu, d_\nu)$.

One may speculate that this is because such sets can be much more complicated than finite unions of intervals; indeed one can have a situation where $\sum_{\nu=1}^{\infty}(d_\nu - c_\nu) < \epsilon$, while the closure of the set $\cup_{\nu=1}^{\infty}(c_\nu, d_\nu)$ is the closed unit interval. In any case it was E. Borel who took the important step of calling

$$\sum_{\nu=1}^{\infty}(d_\nu - c_\nu)$$

the measure of the set

$$\bigcup_{\nu=1}^{\infty}(c_\nu, d_\nu)$$

and extending this measure to a larger class of sets.

Let us also not fail to notice, although Borel and Lebesgue and even Zermelo do not seem to have noticed it, that the above theorem, proved with only a very small preamble and no epilogue, contains the key fact needed in measure theory, namely *countably additivity of interval lengths over intervals*, and without which measure theory would remain only the theory of absolutely convergent series. For let the unit interval (0, 1) be the union of a countable number of pairwise disjoint intervals $I_1, I_2, \ldots$. Any given I_k may be closed, open or semi-open. Let c_k, d_k be the end points of I_k. Then the countable set $\{c_k, d_k, k = 1, 2, \ldots\}$ consisting of end points of the intervals $I_k, k = 1, 2, \ldots$ is a closed set in the interval (0, 1) (since the compliment is the union of open intervals $(c_k, d_k), k = 1, 2, \ldots$), and so by the theorem above as proved by Cantor (sans Zermelo's remark) $f(1) = 1$, i.e., sum of lengths of the intervals, namely $\sum_{k=1}^{\infty}(d_k - c_k) = 1$.

From the very early expositions of measure theory this fact is proved using the Heine–Borel theorem, and despite being the key ingredient of the subject, it is not attributed to Cantor.

3. Volume, Inhalt, Grandeur

Let us see how Cantor defines the metrical size of a set in $\mathbb{R}^n$. It is understood that we know how to calculate the volume of a ball or a cube in $\mathbb{R}^n$, and of finite union of such sets. Cantor defines for any bounded set P in $\mathbb{R}^n$ the volume of P as follows ([1], p. 229, 257): Fix a $\rho > 0$ and let $K(x, \rho)$ denote the sphere of radius ρ with centre $x \in P \cup P'$. Denote the union

$$\bigcup_{x \in P \cup P'} K(x, \rho)$$

of these spheres by

$$M(\rho, P \subset \mathbb{R}^n),$$

or simply by

$$M(\rho, P).$$

Cantor says that $M(\rho, P)$ is a set whose volume may be determined by the usual method of multiple integration. One may object to this since it is not clear that $M(\rho, P)$ is a finite union of balls. But this objection can be easily met as follows: Since ρ is fixed and P is bounded, it is clear (without using Heine–Borel theorem) that finitely many $K(\rho, P)$ are enough to cover $P \cup P'$ and we may replace $M(\rho, P)$ by such a finite union. Write

$$f(\rho, P \subset \mathbb{R}^n) = f(\rho, P) = \int_{M(\rho, P)} dx_1 dx_2 \ldots dx_n,$$

which is the n-dimensional volume of $M(\rho, P)$. For fixed P, f is continuous and decreasing function of ρ. Write

$$I(P \subset \mathbb{R}^n) = I(P) = \lim_{\rho \to 0} f(\rho, P),$$

and call $I(P)$ the volume (or inhalt ([1], p. 229), or grandeur ([1], p. 256)) of the set P with respect to the n-dimensional space $\mathbb{R}^n$. Cantor emphasizes through choice notation and in words that $I(P)$ very much depends on the number n, the dimension of the space $\mathbb{R}^n$ in which P is considered to be imbedded. Thus for the unit square S in $\mathbb{R}^2$, $I(S, \mathbb{R}^3) = 0$ while $I(S, \mathbb{R}^2) = 1$. The positivity of $I(P)$ tells us something about the dimension of the space in which P is imbedded. This may seem like a trivial remark, but we must remember that the question "what is dimension of a set?" was an unresolved foundational question when Cantor was writing, and he had already proved that the notion of cardinality failed to distinguish between sets of different dimension. On the other hand, positivity of $I(P, \mathbb{R}^n)$ implied that while P is a subset of $\mathbb{R}^n$, P can not be contained in a smaller dimensional subspace. The question of dimension mentioned above was resolved only in the early part of the last century through the speculative writings of Poincaré and the work of Brouwer, Lebesgue, Urysohn, Alexandroff, Hausdorff and others ([3]). Hausdorff was a set theorist (he wrote a book on set theory) and it is natural to wonder if Cantor's remarks stimulated him to define his p-dimensional measure and the associated dimension of the set.

Let us proceed. Unless otherwise mentioned $I(P)$ will denote Cantor's n-dimensional volume. Clearly

$$I(P) = I(P \cup P') = I(\overline{P}),$$

where $\overline{P}$ denote the closure of P. $I(P)$ is not always additive over a pair of disjoint sets, for if we take $P =$ rational numbers in (0, 1), and $R = (0, 1) - P$, then

$$I(P \cup R) = I((0, 1)) = 1 < I(P) + I(R) = 2.$$

However, I is additive for sets which are well separated in the sense that they are contained in disjoint open sets. In particular I is additive over closed sets. Cantor is interested in computing $I(P)$ and proves what he calls fundamentalsatz:

$$I(P) = I(P'),$$

whose proof is not too long, followed by a stronger result

$$I(P) = I(P^{\gamma})$$

for any countable ordinal γ. This may seem to follow immediately from the fundamentalsatz, but it is not so, and non-trivial work is needed to prove this for a limit ordinal. We thus see that either

$$\text{either} \quad I(P) = 0 \text{ or } I(P) = I(C),$$

where C is the largest perfect set in P. A measure theoretic-proof of this is easy since $\overline{P} - C$ is countable, but Cantor did not have this theory at his disposal.

For a linear set $P \subset (0, 1)$ Cantor computes $I(P)$ as follows: we know that $I(P) = I(P \cup P')$ and since $P \cup P'$ is closed, $R = (0, 1) - (P \cup P')$ is made of countable number of pairwise disjoint open intervals (c_ν, d_ν), $\nu = 1, 2, 3, \ldots$. Cantor says that it is easily seen that

$$I(P \cup P') = 1 - \sigma,$$

where $\sum_{\nu=1}^{\infty}(d_\nu - c_\nu) = \sigma$. We have seen this already in section 2. Note that for a closed set P, $I(P)$ is the outer Lebesgue measure of P and $1 - \sigma$ is the inner Lebesgue measure of P. *Thus for closed sets P Cantor shows that outer Lebesgue measure of P is equal to the inner Lebesgue measure of P, so that P is Lebesgue measurable, verifying the fact 16 years before Lebesgue measure appeared on the scene.* It is not hard to prove, although Cantor does not prove it, that $I(P)$ is countably additive within the class of closed sets.

It is also interesting to note that with Cantor's inhalt the Banach–Tarski paradox is no more a paradox, since I is not finitely additive.

4. Perfect Sets and their Cardinality, Singular Functions and Measures

Let us see now how Cantor proves that a perfect set without interior has the cardinality of the continiuum, and what consequences it has for measure theory. (Note that if a set has a non-empty interior, it necessarily has the cardinality of the continuum, so only perfect sets without interior need be considered.)

For the special case of perfect $P \subset [0, 1]$ with $I(P) > 0$, he has the following proof which puts to use the function I: ([1], p. 259): Consider the function

$$\phi(x) = I(P \cap [0, x]), 0 \leq x \leq 1.$$

Then ϕ is continuous and assumes all values between 0 and $I(P)$. Further ϕ is constant on the intervals (c_ν, d_ν), $\nu = 1, 2, 3, \ldots$, although constant value can be different for different intervals. Moreover the points c_ν, d_ν, $\nu = 1, 2, 3, \ldots$ all belong to P, so that on P, ϕ assumes all the values between 0 and $I(P)$, which proves that P has the cardinality of the continuum.

We now give Cantor's proof of the fact that any perfect set $P \subset [0, 1]$, not necessarily with $I(P) > 0$, has the cardinality of the continuum ([1], p. 238, [1] p. 252). As mentioned above we may assume that P has no interior. We will also assume without loss of generality that the points 0 and 1 are in P. For the proof we will need the fact, which Cantor proves in the course of his proof, that if A and B are two countable ordered sets without least or greatest elements and such that between any two distinct elements there is a third element, then A and B are order isomorphic in the sense that there is an order preserving bijection between the A and B. Now P is perfect and nowhere dense, so $R = (0, 1) - P$ is a countable union of pairwise disjoint open intervals (c_ν, d_ν), $\nu = 1, 2, 3, \ldots$. Two such intervals can not have a common end points since P has no isolated points. Since P is nowhere dense, between any two distinct intervals (c_ν, d_ν) and (c_λ, d_λ) there is another such interval. Order the intervals by writing $(c_\nu, d_\nu) < (c_\lambda, d_\lambda)$ if $d_\nu < c_\lambda$, equivalently if (c_ν, d_ν) is to the left of (c_λ, d_λ). Clearly the system of intervals (c_ν, d_ν), $\nu = 1, 2, 3, \ldots$ has no least or greatest element since 0 and 1 are in P. Further between any two such intervals there is another such intervals and the system of intervals is countable. Let $Q = \{\phi_1, \phi_2, \phi_3, \ldots\}$ be a countable dense set in $(0, 1)$ with the usual ordering of the real line restricted to Q. Then Q does not admit a least or largest element and between any two elements of Q there is another element of Q. So we can establish an order-preserving bijection between Q and the system of intervals (c_ν, d_ν), $\nu = 1, 2, 3, \ldots$. Let ψ_ν be the element of Q which corresponds to (c_ν, d_ν) under this bijection and write

$$f(x) = \psi_\nu \text{ if } x \in (c_\nu, d_\nu), \nu = 1, 2, 3, \ldots.$$

So that f is a non-decreasing function from $\cup_{\nu=1}^{\infty}(c_\nu, d_\nu)$ to Q. Cantor shows that f extends to a continuous function from [0, 1] to [0, 1] which is obviously non-decreasing and constant on intervals (c_ν, d_ν), $\nu = 1, 2, 3, \ldots$, the constant being ψ_ν on (c_ν, d_ν). On the set $R = \cup_{\nu=1}^{\infty}(c_\nu, d_\nu)$, f assumes countable number of distinct values $\{\psi_1, \psi_2, \psi_3, \ldots\} = Q$, while the image $f[0, 1]$ is all of [0, 1]. We see that $[0, 1] - Q \subset f(P)$ so that P has the cardinality of the continuum. In reality we have $f(P) = [0, 1]$ since the points c_ν, d_ν are all in P and f is continuous so the value of f at c_ν, d_ν is ψ_ν, $\nu = 1, 2, 3, \ldots$.

The well-known Cantor ternary set and Cantor function appear as concrete illustration of the above general construction where the role of Q is played by the dyadic rationals in (0, 1). ([1], p. 255)

Remarks.

(1) Note that as we run over all countable dense subsets of (0, 1), f runs over all continuous non-decreasing functions with distinct constant values on distinct intervals $(c_\nu, d_\nu), \nu = 1, 2, 3, \ldots$. In particular, if $\sum_{\nu=1}^{\infty}(c_\nu, d_\nu) = 1$, so that $I(P) = 1$, then as Q varies over all countable dense subsets of (0, 1), f runs over all non-decreasing continuous singular functions over [0, 1] with points of increase precisely in P.

(2) As Q runs over all countable dense subsets of (0, 1) we obtain all continuous probability measures on [0, 1] whose close support is P, in particular if $I(P) = 0$, we obtain all continuous singular measures on [0, 1] whose closed support is precisely P.

(3) It is clear how this method can be used to show that any two perfect nowhere dense sets are homeomorphic.

(4) The method of construction is used by A. Denjoy to construct a homeomorphism of the circle group S^1 with any irrational rotation number α whose invariant measure is supported on a given perfect set of S^1. ([2], p. 81)

5. The Cantor's Integral

Cantor also defined an integral over an arbitrary set which coincides, for closed sets, for the functions he had in mind, with the Lebesgue integral.

If ϕ is an arbitrary *absolutely* integrable function on $\mathbb{R}^n$, and P is an arbitrary bounded set of $\mathbb{R}^n$, and if $M(\rho, P \subset \mathbb{R}^n) = M(\rho, P)$ be as defined above, then the integral

$$\int_{M(\rho,P)} \phi(x_1, x_2, \ldots, x_n) dx_1 dx_2 \ldots dx_n$$

is a continuous function of ρ so the limit of this function as ρ tends to 0 is a definite number which depends on ϕ. Cantor denotes this number by $I(\phi(x_1, x_2, \ldots, x_n), P)$ and observes that $I(P) = I(1, P)$. Note that the function f is not required to be bounded or continuous, but only absolutely integrable over $\mathbb{R}^n$ which should be interpreted to mean that unconditional improper integral of ϕ exists. For close sets, more generally for sets P with inhalt of $\overline{P} - P$ zero, $I(\phi(x_1, x_2, \ldots, x_n), P)$ agrees with the Lebesgue integral of ϕ over P. Proof of this will need a form of monotone convergence theorem.

Cantor's integral does not widen the class of functions which can be integrated, as is the case with Lebesgue integral, but it widens the class of sets on which integrals can be taken, and in particular one could speak of integral over a closed set without interior.

Cantor does not put this integral to use or derive any of its properties but in the definition the word 'unbedingt' (which means absolute or absolutely) appears in italics, and one may wonder if Cantor had some deep feeling that a successful theory of integral will integrate only all those functions whose absolute values are integrable.

6. The Equality $I(P) = I(\overline{P})$

In the light of the success of the theory of countably additive measures and integrals, the property $I(P) = I(\overline{P})$ of Cantor's 'inhalt' and the resulting non-additivity of I (except over well-separated sets) is a severe drawback. However, viewed differently the property

$$I(P) = I(\overline{P})$$

encodes a deep measure theoretic phenomenon connecting finitely additive measures with countably additive measures. Consider the set J of rational numbers between 0 and 1, and we denote this *interval of rational numbers* by $< 0, 1 >$. For $0 < a < b < 1$, $< a, b >$ will mean the set of *rationals* x, $a < x < b$.

For $a, b \in J, a < b$, it seems natural to define $b - a$ as the length of the interval $< a, b >$, and we denote this by

$L(< a, b >)$. If f is a uniformly continuous function on $< 0, 1 >$ then one can define the 'Riemann Integral' of f on $< a, b >$ in the usual manner and observe that

$$\int_{<0,1>} f dL = \int_0^1 \overline{f}(x)dx,$$

where the right hand side is the usual Riemann integral of the continuous extension $\overline{f}$ of f to [0, 1].

When J is viewed as a subset of the unit interval (0, 1) of real numbers, we see that

$$L(< a, b >) = b - a = I(< a, b >) = I(\overline{< a, b >}),$$

where $\overline{< a, b >}$ is the closure of $< a, b >$ in (0, 1). Note that L extends to a finitely additive measure on the algebra on J generated by intervals of the form $< a, b >$ while I is countably additive on the algebra generated by intervals in (0, 1) and extends to a countably additive measure on (0, 1).

We consider two more examples before stating the general phenomenon. Consider the real Hilbert space l^2 of square summable-sequences of real numbers. For each n, we view $\mathbb{R}^n$ as a subspace of l^2, and consider the countably additive measure on $\mathbb{R}^n$ given by the density

$$\frac{1}{(2\pi)^{\frac{n}{2}}} \exp(-x_1^2 - x_2^2 - \cdots - x_n^2).$$

As n varies we get a system of measures whose 'projective limit' L on l^2 is only finitely additive. But, as is well known, there is a Banach space B in which l^2 is densely embedded with continuous injection map, and the Banach space B admits a countably additive measure m such that for any set $P \subset l^2$ for which L is defined we have

$$L(P) = m(\overline{P}),$$

where $\overline{P}$ is the closure of P in B, moreover $m(l^2) = 0$.

Consider again the natural density d defined on subsets of integers (or rather on a class of subsets). If f is a uniformly almost periodic function on the integers $\mathbb{Z}$, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=-N}^{N} f(k)$$

exists, it is called Bohr mean of f, and may be viewed as an integral of f with respect to the finitely additive measure d defined on the subsets of $\mathbb{Z}$ which admit natural density. We know, with Bochner, that $\mathbb{Z}$ admits a compactification $\mathbb{Z}$, called the Bohr compactification, with a normalized Haar measure m such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=-N}^{N} f(k) = \int_{\mathbb{Z}} \overline{f} dm,$$

where $\overline{f}$ is the continuous extension of f to $\mathbb{Z}$. If $A \subset \mathbb{Z}$ has natural density then we know that

$$d(A) = m(\overline{A}),$$

although $m(Z) = 0$. ([5])

Thus the deep phenomenon that the property $I(P) = I(\overline{P})$ of Cantor's inhalt encodes may be stated as follows: *if there is a finitely additive measure Λ on a certain algebra of subsets of a set X, then with good chance there is a completion or compactification Y of X in which X is densely embedded and Y admits a countably additive measure m such that*

$$\Lambda(A) = m(\overline{A}),$$

for any set for which Λ is defined.

7. Cantor's Inhalt and Jordan Content

Let us briefly recall how Jordan content (also due to Peano) of a bounded subset P of $\mathbb{R}^n$ is defined. One forms a grid Π of n-dimensional cubes with sides parallel to the natural $(n-1)$ dimensional subspaces, and writes $\overline{J}(P, \Pi)$ for the sum of volumes of those cubes of the grid which intersect P, and $\underline{J}(P, \Pi)$ for the sum of volumes of the cubes which are subsets of P. The lower bound of $\overline{J}(P, \Pi)$ taken over all grids Π is called the upper Jordan content of P and denoted by $\overline{J}(P)$, while the upper bound of $\underline{J}(P, \Pi)$, taken over all Π, is called the lower Jordan content of P, and denoted by $\underline{J}(P)$. Clearly

$$\underline{J}(P) \leq \overline{J}(P),$$

and J is said to have Jordan content if

$$\underline{J}(P) = \overline{J}(P).$$

Subsets of the unit cube in $\mathbb{R}^n$ which have Jordan content form an algebra, and a set P has Jordan content if and only of 1_P is Riemann integrable. It is clear that for any set P, the outer Jordan content of P is same as the inhalt of P, and that closed sets without interior but with positive $I(P)$, sets of interest to Cantor, have no Jordan content. Further, the sets

$\{P : I(\overline{P} - P) = 0\}$ form a ring much bigger than sets having Jordan content and I is additive on these sets. It is therefore not correct to suggest, with Lebesgue ([4], p. 37) that Jordan's exposition simplifies and completes the definition given by Cantor. Rather Cantor's and Jordan's ideas were revised and modified in a very important way and fully developed in the work of Borel and Lebesgue.

Whether Cantor's comments on dimension and inhalt, his definition of the integral over an arbitrary set, and the equality $I(P) = I(\overline{P})$ should be credited with insights suggested here is a matter of opinion.

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A Completeness Axiom for the Real Numbers that is Useful in Calculus

Bernd S. W. Schröder
Program of Mathematics & Statistics,
Louisiana Tech University, Ruston, LA 71272
schroder@coes.LaTech.edu

This paper presents a simple characterization of the divergence of sequences of real numbers. The criterion is equivalent to the completeness of the real numbers. It can be used in an intuitive, yet rigorous, presentation of limits and it facilitates an implicit review of some precalculus.

As the conceptual foundation for derivatives and integrals, the limit is the start of calculus. This start can be challenging for students. To the author, the key conceptual adjustment is that static computations from algebra are now replaced with considerations that ask "What happens if ⟨a number gets close to another number?⟩, ⟨a number grows beyond all bounds?⟩, etc." History shows that this adjustment is not trivial at all. The mathematicians in ancient Greece almost, but not quite, got there. Leibnitz and Newton introduced calculus, but the rigorous foundation as we know it was missing until the 1800s[1].

Aside from this significant conceptual leap, instruction may need to take into account that a good number of freshmen could use some reinforcement of the very algebra and trigonometry on which calculus computations rely. This realization is seemingly at odds with the entirely reasonable demand that calculus must not be watered down under any circumstances. Since the start of calculus reform in the late 1980s, this has been a point of lively discussion. Intuitive presentations of the limit as well as the use of $\varepsilon - \delta$ proofs in early calculus have been vigorously advocated. Yet this author is unaware of any attempt to tackle the problem via the well-known characterization "$\lim_{x\to a} f(x) = L$ if and only if for all sequences $a_k \to a$ we have $\lim_{k\to\infty} f(a_k) = L$." This paper describes how an earlier introduction of limits of sequences together with a presentation of limits of functions through sequences can lead to a better conceptual and formal grasp on the notion of convergence. This content shift does not add time to the curriculum and the earlier treatment of sequences can also be used to solidify key

[1] Not to mention that Leibnitz's original idea of "differentials" was only formalized in Nonstandard Analysis in the 1960s.

elements from precalculus by exhibiting them in a new context rather than regurgitating them once again.

Presently, many students leave the calculus sequence with an intuitive understanding of convergence and divergence that allows them to explain pictorially why limits such as $\lim_{x\to 1} \frac{x-1}{|x-1|}$ and $\lim_{x\to\infty} \sin(x)$ do not exist. (We will use these two limits as standard examples throughout.) A formally correct proof cannot be expected with the tools available in the currently canonical presentation of calculus. While the author contends that the ability to write formal proofs is not and should not be the foremost goal of most calculus classes, it would be nice to alleviate the insecurities caused by pictorial arguments. Even if they contain the heart of a formal proof, pictorial arguments are by nature intuitive and imprecise. This means students can rightly find them unmathematical. Mathematics takes intuition and puts it on a rigorous, understandable foundation. This cannot be done at this stage and consequently students will be insecure. In the author's experience, answers for test questions asking why limits $\lim_{x\to 1} \frac{x-1}{|x-1|}$ and $\lim_{x\to\infty} \sin(x)$ do not exist are weaker than answers for straight limit computations. Omitting visual arguments is not an option, but there is an intuitive, yet rigorous, counterpart. Translating the pictures into a rigorous argument can be seen as a contribution to implementing recommendations 2 (develop mathematical habits of mind, use multiple approaches, communicate with clarity and coherence) and 3 (present key ideas and concepts from a variety of perspectives) of [3].

Sections 1 to 3 present definitions and results in the order in which they could be presented in a class. Section 4 shows some delayed benefits from using sequences in the presentation of continuity and differentiability. Section 5 discusses how convergence of sequences can be used to review precalculus and Section 6 shows connections to engineering and science. We conclude in Section 7 with a discussion of timing and possible pitfalls.

1. Presentation of Sequences

It is possible to start calculus with an intuitive explanation of convergence of sequences given by an algebraic expression. Convergence of sequences is part of calculus, and there is no logical prerequisite that forbids moving it to the start. Moreover, an intuitive explanation of convergence of sequences is implicit in early calculus when convergence of functions at $\pm\infty$ is discussed. To make things completely rigorous, some time can be spent on the "$\varepsilon - K$" definition of convergence of sequences: $\lim_{k\to\infty} a_k = A$ if and only if for every $\varepsilon > 0$ there is a K such that for all $k > K$ we have that $|a_k - A| < \varepsilon$.

For deeper intuitive and formal investigation of limits, introduce subsequences (see Section 5 for more details) and the divergence criterion in Theorem 1.1. In freshman calculus, Theorem 1.1 would typically be presented without proof.

Theorem 1.1. (Divergence criterion for sequences of real numbers). *A sequence $\{a_k\}_{k=1}^{\infty}$ of real numbers diverges if and only if*

(1) There is a subsequence $\{a_{k_m}\}_{m=1}^{\infty}$ such that $\lim_{m\to\infty} a_{k_m} = \infty$ or $\lim_{m\to\infty} a_{k_m} = -\infty$, or
(2) There are two subsequences $\{a_{i_m}\}_{m=1}^{\infty}$ and $\{a_{j_n}\}_{n=1}^{\infty}$ such that $\lim_{m\to\infty} a_{i_m}$ and $\lim_{n\to\infty} a_{j_n}$ both exist, but are not equal.

Proof. Routine $\varepsilon - K$ arguments show that both criterion 1 and criterion 2 imply that the sequence $\{a_k\}_{k=1}^{\infty}$ diverges.

To prove that divergence implies 1 and 2, we use the Bolzano–Weierstrass theorem. Let the sequence $\{a_k\}_{k=1}^{\infty}$ be divergent. If 1 holds we are done. Otherwise the sequence $\{a_k\}_{k=1}^{\infty}$ is bounded. By the Bolzano–Weierstrass theorem there is a subsequence $\{a_{i_m}\}_{m=1}^{\infty}$ that converges to a limit M. Since the sequence $\{a_k\}_{k=1}^{\infty}$ diverges, there is an $\varepsilon > 0$ and a subsequence $\{a_{k_l}\}_{l=1}^{\infty}$ such that for all l we have that $\left|a_{k_l} - M\right| > \varepsilon$. Since $\{a_{k_l}\}_{l=1}^{\infty}$ is bounded, it has by the Bolzano–Weierstrass theorem a subsequence with a limit $L \neq M$. Said subsequence is also a subsequence of $\{a_k\}_{k=1}^{\infty}$. □

Although the $\varepsilon - K$ arguments in the first part of the proof are accessible in freshman calculus, they are quite tedious at this stage. The second part of the proof is not accessible in freshman calculus. Fortunately, omitting the proof (or at least the second part) is not a loss of rigor. This is because the divergence criterion for sequences can be awarded the status of an axiom. Recall some of the manifold equivalent completeness criteria/axioms for the real numbers. We omit the proof, as it is standard fundamental analysis.

Theorem 1.2. *The following are equivalent for the real numbers.*

(1) Every Cauchy sequence has a limit.
(2) Every subset that is bounded above has a lowest upper bound.

(3) *Monotone Convergence Theorem. Every monotone, bounded sequence converges.*

(4) *Nested Interval Principle. If $[a_n, b_n]$ is a sequence of intervals such that $[a_n, b_n] \supseteq [a_{n+1}, b_{n+1}]$, then there is a point c such that $c \in [a_n, b_n]$ for all n.*

(5) *The Bolzano–Weierstrass Theorem. Every bounded sequence has a convergent subsequence.*

Typically only 1 or 2 are awarded the status of an axiom in analysis. Yet the equivalence of all criteria shows that indeed any of them could be used as an axiom. In an elementary proof-based analysis class, it is sensible to use 1 or 2 as an axiom and to derive the other criteria. Either of 1 or 2 may have been proved in an earlier construction of the real number system, and proving the rest is good training in proof writing.

Few of the criteria in Theorem 1.2 are presented explicitly in early calculus. Limits of Cauchy sequences and the Bolzano–Weierstrass theorem are not presented. Existence of lowest upper bound is mentioned at best and the nested interval principle is taken for granted when algorithms such as the bisection method are executed. Solely the Monotone Convergence Theorem is introduced explicitly and that usually in a supporting role when limits of recursive sequences are computed. There is good sense in omitting these criteria from freshman calculus. They are all rather abstract and, in the author's opinion, truly most appropriate for proof-based classes.

In comparison, Theorem 1.1 is very intuitive. Just what else could happen when a sequence fails to converge? Theorem 1.1 lists the two obvious "failure modes" (see Section 6 for more on this choice of language) and then asserts that failure cannot happen in any other fashion. Fortunately, Theorem 1.1 is equivalent to the completeness criteria for the real numbers, and hence the author suggests to elevate it to the status of an axiom for freshman calculus with no proofs presented.

Theorem 1.3. *The divergence criterion in Theorem 1.1 is equivalent to all statements in Theorem 1.2.*

Proof. The proof of Theorem 1.1 shows that it follows from the Bolzano–Weierstrass theorem.

The divergence criterion also easily implies the Bolzano–Weierstrass theorem. Let $\{b_k\}_{k=1}^{\infty}$ be a bounded sequence of real numbers. If $\{b_k\}_{k=1}^{\infty}$ converges, there is nothing to prove. Otherwise, by part 2 of Theorem 1.1 it has two convergent subsequences. $\square$

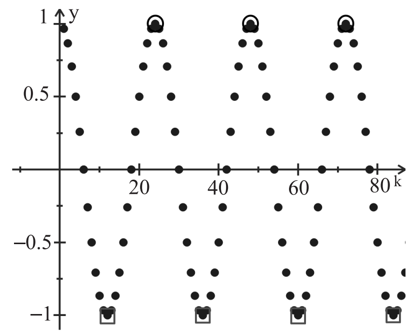

Figure 1. The sequence $\left\{\cos\left(\frac{k\pi}{12}\right)\right\}_{k=1}^{\infty}$ diverges, because several subsequences have different limits. In the picture, subsequences with $k_m = 24m$ and $k_n = 12 + 24n$ are marked with circles and squares, respectively. Their limits are 1 and -1, respectively

After using subsequences and the divergence criterion to analyze some divergent sequences, such as for example $\left\{\cos\left(\frac{k\pi}{12}\right)\right\}_{k=1}^{\infty}$ (see Figure 1, the class could move on to functions. Recursive sequences, including an intuitive proof of the monotone convergence theorem based on Theorem 1.1, could be weaved in here or postponed.

2. Presentation of Limits of Functions – Limits at Infinity

Limits of functions at $\pm\infty$ are computationally very similar to limits of sequences. Hence the computational part can be covered efficiently. The idea of an asymptote adds more visualization to the investigation. The formal definition of limits at infinity could be given through sequences.

Definition 2.1. *Let f be a function defined on an interval (s, ∞) and let L be a real number. Then we say that f converges to L as x goes to infinity, in symbols $\lim_{x\to\infty} f(x) = L$, if and only if for every sequence $a_k \to \infty$ we have that $\lim_{k\to\infty} f(a_k) = L$.*

The key advantage of this definition is its logical simplicity. There is one universal quantification instead of the usual three nested quantifications and we are not working with inequalities. Of course the complexities are "hidden" by shifting them to the underlying sequences, but this nesting of concepts merely allows for more effective use of the brain. It is not a loss of rigor. Moreover, for problems such as showing that $\lim_{x\to\infty} \sin(x)$ does not exist, students now only have to find one input sequence along which the function fails to converge or, since the function is bounded, two input sequences that go to infinity and along which the function takes different limits.

At this stage one can state (and may prove) a divergence criterion for functions at infinity. Analogous to Theorem 1.1, $\lim_{x\to\infty} f(x)$ does not exist if and only if there is a sequence $a_k \to \infty$ so that $f(a_k)$ goes to plus or minus infinity or there are two sequences $a_k, b_k \to \infty$ so that $\lim_{k\to\infty} f(a_k) \neq \lim_{k\to\infty} f(b_k)$. For the sine function, the visual observation that the limit cannot exist, "because the maxima and minima stay 2 units apart" can then be formalized by exhibiting the input sequences $\left\{\frac{\pi}{2}+2\pi k\right\}_{k=0}^{\infty}$ and $\left\{\frac{3\pi}{2}+2\pi k\right\}_{k=0}^{\infty}$ (also see Figure 2) which both go to infinity and yet the sine is equal to 1 along the former and equal to -1 along the latter.

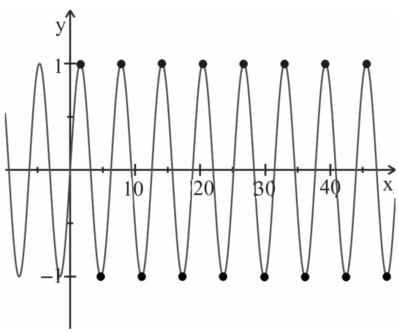

Figure 2. The sine function in an interval chosen to exhibit long-term behavior. The sequences used to establish divergence are marked with solid dots

In a more theoretical class the connection between Definition 2.1 and an $\varepsilon - M$ definition of convergence could be explored.

3. Presentation of Limits of Functions – Limits at a Point

After limits at infinity, limits at a point can be defined in similar fashion.

Definition 3.1. *Let f be a function defined on an interval around the point a, but not necessarily at a, and let L be a real number. Then we say that f converges to L as x goes to a, in symbols $\lim_{x\to a} f(x) = L$, if and only if for every sequence $a_k \to a$ we have that $\lim_{k\to\infty} f(a_k) = L$.*

The pattern of the definition is similar to Definition 2.1, so there is no large jump here. Limit laws and algebraic computations can be presented just as always. For divergence at a point, again the definition is simpler and a divergence criterion could be presented (and possibly proved). Analogous to Theorem 1.1, $\lim_{x\to a} f(x)$ does not exist if and only if there is a sequence $a_k \to a$ so that $f(a_k)$ goes to plus or minus infinity or there are two sequences $a_k, b_k \to a$ so that $\lim_{k\to\infty} f(a_k) \neq \lim_{k\to\infty} f(b_k)$. Note that this is the third divergence criterion we encounter and they are all very similar. This repetition should solidify the intuition that divergence means either growth beyond all bounds or some type of branching or oscillation.

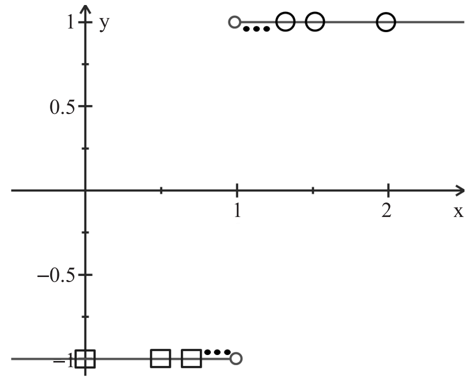

Figure 3. The non-existence of a limit of $f(x) = \frac{x-1}{|x-1|}$ at $a = 1$ is detected/proved by considering the sequences $\left\{f\left(1 \pm \frac{1}{k}\right)\right\}_{k=1}^{\infty}$ which have different limits. The corresponding points $(a_k,\ f(a_k))$ are marked with circles and squares respectively

In this fashion the observation that $\lim_{x\to 1} \frac{x-1}{|x-1|}$ does not exist, because there is a jump at $a = 1$ can be translated into a formally correct argument: The function $f(x) = \frac{x-1}{|x-1|}$ diverges at $a = 1$, because (also see Figure 3)

$$\lim_{k\to\infty} f\left(1+\frac{1}{k}\right) = 1 \neq -1 = \lim_{k\to\infty} f\left(1-\frac{1}{k}\right).$$

Adding and subtracting $\frac{1}{k}$ can become somewhat of a standard trick when detecting jumps. Investigation of functions such as $g(x) = \sin\left(\frac{1}{x}\right)$ (see Figure 4) should ensure that this trick will not be applied mindlessly.

As for limits at infinity, a more theoretical class can explore the connection to the usual $\varepsilon - \delta$ definition. At this stage the author must admit that he considers Definition 3.1 is superior to the $\varepsilon - \delta$ definition. The language and the intuition behind limits reflect motion ("limit as x *goes to* . . . ," "convergence *to* . . . ," etc.), which is exactly what is stated in Definition 3.1. The images of all input sequences that *go to a* must *go to L*.

4. Effects on the Presentation of Continuity and Derivatives

The presentation of continuity in calculus sometimes provides pictures of discontinuities by oscillation, but the tools at hand do not allow a precise introduction and analysis of these discontinuities. Via sequences, a function f has a discontinuity by oscillation at a if and only if two sequences $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$ approach *a from the same side* and $\lim_{k\to\infty} f(a_k) \neq \lim_{k\to\infty} f(b_k)$ (this is also visualized in Figure 4). Moreover, with a simple case analysis it can now be shown that any discontinuity of a function is either removable, a jump, an infinite discontinuity (these three are usually defined in calculus) or a discontinuity by oscillation. Thus working with sequences gives access to *all* the "failure modes" of continuity.

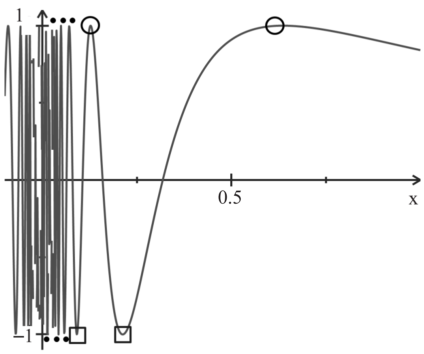

Figure 4. Adding and subtracting $\frac{1}{k}$ is not a cure all to prove non-existence of limits, as $g(x) = \sin\left(\frac{1}{x}\right)$ shows. For this function, input sequences such as $a_k = \frac{1}{\frac{\pi}{2}+2\pi k}$ (points $(a_k, f(a_k))$ marked with circles) and $b_k = \frac{1}{\frac{3\pi}{2}+2\pi k}$ (points $(b_k, f(b_k))$ marked with squares) can be used to show divergence at $a = 0$. The function g is also a nice prototype for discontinuities by oscillation

The standard approach of working with $\pm\frac{1}{k}$ to detect jumps can also help solidify arguments why a given function is not differentiable at a point because of a corner. Derivatives are computed as limits $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ and the function is not differentiable when the limit does not exist. In addition to intuitive arguments about how there cannot be a tangent line at a corner, analysis of the limits $\lim_{k\to\infty} \frac{f\left(x\pm\frac{1}{k}\right)-f(x)}{\pm\frac{1}{k}}$ can provide a formal justification why, for example, $f(x) = |x|$ is not differentiable at $a = 0$.

5. How Subsequences Relate to Precalculus

Having established where and how sequences can fit into early calculus, it is now worth considering some benefits beyond the deeper investigation of convergence. The primary benefits are a precalculus review "in context," discussed in this section, and connections to engineering and science, discussed in the next.

Arguments as in Figure 1, and subsequent similar arguments for functions, can only be given if students have reached a certain level of comfort with subsequences. This practice with subsequences also serves as a reinforcement of fundamentals from precalculus.

On a symbolic level, the computation of the expression a_{k_n} from a_k and k_n is nothing but the composition of functions with different notation. Sufficiently complex compositions, for example, investigating $\lim_{k\to\infty} f\left(x \pm \frac{1}{k}\right)$ for a reasonably complicated function or even a difference quotient such as $\lim_{k\to\infty} \frac{f\left(x\pm\frac{1}{k}\right)-f(x)}{\frac{1}{k}}$, also provide a rapid review of the laws of algebra in a "real" context.

Conversely, finding subsequences of sequences such as $\left\{\cos\left(\frac{k\pi}{12}\right)\right\}_{k=1}^{\infty}$ requires the student to recognize the expression as a composition of the cosine function with the expression $\frac{k\pi}{12}$, which is preparation for the recognitions needed for the chain rule in differential calculus.

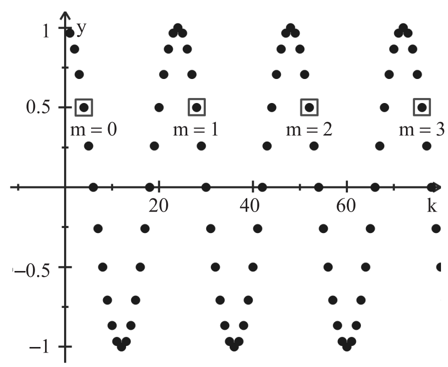

Figure 5. The sequence $\left\{\cos\left(\frac{k\pi}{12}\right)\right\}_{k=1}^{\infty}$ with the subsequence $\{a_{4+24m}\}_{m=0}^{\infty}$ that converges to $\frac{1}{2}$ marked by boxed dots

Aside from showing composition in a slightly different light and preparing for divergence proofs, an exercise such as "find a subsequence of $\left\{\cos\left(\frac{k\pi}{12}\right)\right\}_{k=1}^{\infty}$ with limit $\frac{1}{2}$," also requires students to visualize sequences and subsequences (see Figure 5) and to recall the standard values of the trigonometric functions. The visualization of sequences and subsequences as in Figure 5 is essentially plotting points and visually identifying a pattern within a sequence. Note that many of our students will analyze sequential data in science and engineering classes. Similar analysis in a mathematics class and the subsequent analysis of the interpolating functions when limits at infinity are discussed introduce or reinforce the idea of sampling. This idea is important in applications as well as when choosing the right window on your graphing calculator.

Standard values of trigonometric functions are a well-known, annoying and unnecessary stumbling block later in calculus. Proper reinforcement of the values by additional training, such as with subsequences, is the only remedy the author can think of.

6. Connecting the Approach in Calculus to the Approach in Science and Engineering

Talking about sampling in the previous section immediately leads the author to possible connections to engineering and science. Most students take calculus because they will need it in an applied context such as science or engineering. These students need the thought processes of mathematics as well as the computational abilities. The better we connect mathematics content to engineering and the sciences, the less isolated mathematics will appear to students and the more successful students will be applying mathematics in their own fields. Connections to other fields have been advocated in recommendation 4 of [3], the usefulness of connections to keep knowledge from going inert has been demonstrated in [2] and curriculum integration has been practiced in engineering for a while (see [4] for example).

Though it may not look terribly impressive, a small adjustment of language or a parallel use of terms that students hear in engineering and science can help make mental connections between mathematical thought/analysis and the approaches of engineering and science. For example, while we "plot points" in mathematics classes, engineers and scientists "sample time series." Earlier, the author wrote of "failure modes" of convergence when analyzing divergence. Understanding how an experiment/design could fail allows the designer to come up with a better one. This is why in applications the question "How could it break?" is asked frequently. Many mathematical characterizations, including Theorem 1.1 and the similar characterizations of divergence here, are exactly such characterizations of "failure." Introducing this point-of-view can help put students at ease. It also demonstrates the fact that mathematics is intrinsic to any scientific approach.

Similarly, the characterization of limits of functions via sequences reflects the scientific method. A scientific statement can only be considered validated if the results of all experiments are consistent with the statement. A verifiable, reproducible experiment that is at odds with the statement indicates that the statement is not true. Essentially, every sequence we use to "test" the limit is an experiment and only if all experiments are consistent does the limit exist. Again we demonstrate the close link between mathematics and students' chosen fields on a much more fundamental level than just being a computational tool.

7. The Timing Issue and a Possible Pitfall

Any adjustment to the presentation of calculus material must also be evaluated based on its effect on the remainder of the sequence. Extra time spent on one topic is necessarily taken from another. The adjustment suggested here should have no, or at most minimal, effects on the times spent on other topics in calculus.

Limits of functions at infinity and an introduction to sequences are already part of the calculus sequence. Moving the treatment of sequences to the start of the course will merely shift topics back a few days until the time when sequences would normally be covered. At that point the time would be "regained." The suggested approach adds a treatment of subsequences to the course, which may ultimately add a class day. However, this day might be saved by moving more rapidly through limits of functions at infinity, since at least the computations are nothing new. Even if a small amount of time is added here, the demonstrated gains in rigor seem worth it to this author. Mathematics is a discipline that logically derives conclusions from accepted first principles (axioms). The suggested approach allows to present the foundation of calculus in this fashion while enhancing the benefits of an intuitive introduction. Moreover, it shows that rigor and proofs are nothing but structured, unambiguous ways to validate and communicate intuition. Aside from the shifting of content, the only other addition is that some examples that used to be graphical only (jump discontinuities, corners, etc.) can now also be treated rigorously, which will affect a few minutes of some later classes.

Since the author has started using this approach he has observed only one main pitfall. On a test covering limits, a small number of students will try to compute limits at infinity for every limit problem. That is, they mechanically expand with the reciprocal of the highest power of x, etc., even when computing limits at a point. While this can be ascribed to these limits being presented first (and twice), it is not clear to the author if another presentation would be more beneficial for these students. This type of performance reveals an overly schematic approach to mathematics (memorizing problem types) and in some cases it was a consequence of excessive absences at exactly the wrong time. Some students also simply misfired and recovered easily on subsequent exams. So, this pitfall can indicate an underlying problem in the student's approach to learning, and it can then serve as a warning shot from which the student can still recover.

Other than that, it must be said that the analysis of convergence and divergence with sequences is challenging because of its rigorous nature as well as because it can reveal difficulties with visualization and prerequisite topics. Among the chief difficulties in first solving a problem as mentioned in Section 5 was to recall the shape of the cosine function and the values for which the cosine is equal to $\frac{1}{2}$. This however, is an encouragement to use this approach rather than the opposite. Students will need these types of fundamentals throughout their calculus and college careers, so problems are best detected early and remedied by training "in context."

To conclude on an upbeat note, it is encouraging that freshmen can give solid proofs of non-existence of limits on tests and connect these proofs to the appropriate pictures.

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BERND S. W. SCHRÖDER earned a Ph. D. in probability theory and harmonic analysis at Kansas State University in 1992. His main field of mathematical research now is the theory of ordered sets, but he has also done work in graph theory, stochastic processes, and computer science. Since 1997 he has been active in engineering and science education, with significant contributions to the design of the six-course mathematics sequence in Louisiana Tech University's engineering and science curricula and to the sophomore year in the integrated engineering curriculum. He currently serves as the Edmondson/Crump Professor and program chair of the Program of Mathematics and Statistics at Louisiana Tech University.

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Nick Papamichael, University of Cyprus
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Pietro Poggi–Corradini, Kansas State University
Title: Mapping properties of analytic functions on the disk

William T. Ross, University of Richmond
David Shoikhet, Galilee Research Center

Michael Stessin, S.U.N.Y., Albany
Title: Nevanlinna counting currents in hyperconvex domains

Jie Xiao, Memorial University of Newfoundland
Title: Holomorphic Q Classes, Five Years Later

Michel Zinsmeister, Universite d'Orleans
Title: On the product of a BMO and an H^1 function

Nina Zorboska, University of Manitoba
Title: Toeplitz Operators on Analytic Besov Spaces

Financial Support: The organizers reserve limited funds to cover part of the expenses of some invited speakers and participants. Young analysts and students will be given priority.

Proceedings of the Conference:

- The organizing committee is making arrangements for the publication of the proceedings of the conference. The editorial board for this publication includes A. Carbery, University of Edinburgh
- P. Duren, University of Michigan
- D. Khavinson, University of Arkansas

For details: http://web.auth.gr/comhar/index1.html

Geometry and Representation Theory: A Conference in Honor of George Lusztig

May 30–June 2
M.I.T., Cambridge
Massachusetts

For details:
http://math.mit.edu/conferences/lusztig60/index.html

The International Summer School in Several Complex Variables

Szczyrk, Poland
(June 19–23)

Description: We wish to bring together the leading specialists and the young researchers working in pluripotential theory, complex dynamics, complex approximation theory, invariant distances and geometrical aspects of complex analysis. We plan four 4 hours long courses, delivered by leading specialists, introducing to the recent developments in algebraic geometry (given by X. X. Chen (Madison)), multivariable complex dynamics (E. Bedford (Bloomington)), invariant distances (P. Pflug (Oldenburg)) and approximation theory (W. Pleüniak (Kraków)).

Scientific Committee:

E. Bedford (Bloomington)
Cegrell (Umea)
E. Chirka (Moscow)
F. Forstneric (Ljubljana)
P. Guan (Montreal)
T. Ohsawa (Nagoya)
P. Pflug (Oldenburg))
J. Siciak (Kraków)
A. Zeriahi (Toulouse)

Invited Speakers:

T. Bloom (Toronto)
Cegrell (Umea)
Bo–Yong Chen (Shanghai)
D. Coman (Syracuse)
E. Chirka (Moscow)
Dinh Tien–Cuong (Paris)
F. Forstneric (Ljubljana)
V. Guedj (Toulouse)
N. Levenberg (Bloomington)
E. Ligocka (Warszawa)
A. Nicoara (Harvard)
Ohsawa (Nagoya)
E. Poletsky (Syracuse)
G. Tian (Princeton)
A. Zeriahi (Toulouse)

Information and Fee: We estimate the conference fee (including hotel and meals) at 300–320 euro. For further information please contact: E-mail: `scv2006@im.uj.edu.pl`. Also see `http://www.im.uj.edu.pl/scv2006/`

2006 International Conference on Topology and its Applications

Aegion, Greece
(June 23–26)

Description: All areas of Topology and its Applications are included (especially General topology, Applied general topology, Set-theoretic topology, Geometric topology, Topological groups, Dimension theory, Dynamical systems and Continua theory, Computational topology, History of topology).

Deadlines: For the participation form is May 31, 2006. For the abstract form is May 31, 2006. For the accommodation form is March 31, 2006.

Organizing Committee:

S. D. Iliadis (University of Patras) Chairman
D. N. Georgiou (University of Patras)
V. M. Costaras (Aegion)
C. G. Fotopoulos (Aegion)
C. A. Vasilakopoulos (Aegion)

Proceedings: A special issue of the journal Topology and its applications will be devoted to the Proceedings of the 2006 International conference on topology and its applications.

The usual standards of Topology and its applications will be applied.

The submission deadline is October 30, 2006. Instruction for authors: follow the instructions in Topology and its applications (See the Website):

`http://authors.elsevier.com/JournalDetail.html?`
PubID=505624&Precis=&popup=

Each submitted paper should not exceed 20 pages.

Electronic copies of the papers (files: yourname.tex, yourname.ps) must be sent to: `aegion@math.upatras.gr`

Information: `http://www.math.upatras.gr/ aegion/`;

21st International Conference on Operator Theory

The 21st Conference in Operator Theory, organized jointly in Timisoara by the Institute of Mathematics of the Romanian Academy and the West University in Timisoara, will take place between June 29 and July 4, 2006, in Timisoara, Romania.

The Editors of the Journal of Operator Theory: W. B. Arveson, K. R. Davidson, N. K. Nikolskii, S. Stratila and F. H. Vasilescu, constitute the steering committee of the conference.

Theme: The conference is devoted to operator theory, operator algebras and their applications (differential operators, complex functions, mathematical physics, matrix analysis, system theory, etc.).

Registration: The deadline is March 30, 2006. You can either register online, or send an E-mail message to:
`ot@imar.ro` or `ot@theta.ro`.

Programme: The Conference will be held in the main building of the University of Timisoara. Two daily sessions, morning and afternoon, are planned. There will be plenary lectures of 40 min. and communications of 30 min. and 20 min. Subject to the number of registered participants, several parallel sections will be organized.

Abstracts: An abstract of maximum 20 lines of the corresponding lecture/communication should be submitted online on the Web at

`http://atlas-conferences.com/cgi-bin/abstract/submit/case-01`

Abstracts can be viewed at

`http://atlas-conferences.com/cgi-bin/abstract/case-01`

Alternatively, abstracts can be sent by E-mail to `ot@imar.ro` or `ot@theta.ro`. The deadline for receiving abstracts is June 15th, 2006.

Contact:

OT21, Institute of Mathematics,
P. O. Box 1-764,
014700 Bucharest, Romania.

International Conference on Analytic Topology and its Applications

10–14, July 2006
Rotorua
New Zealand

Description: The main goal of this conference is to bring together a group of researchers from around the world, who are working at the inter-face between topology and analysis, to discuss the recent developments and future directions of Analytic Topology.

There will be a series of one hour plenary talks and 20 minute contributed talks on various topics.

Organisers: Jiling Cao and Warren B. Moors

Plenary Speakers:

Jonathan M. Borwein (University of Dalhousie, Canada)
Bernardo Cascales (University of Murcia, Spain)
Petar Kenderov (Bulgarian Academy of Sciences, Bulgaria)
Tsugunori Nogura (Ehime University, Japan)
Roman Pol (Warsaw University, Poland)
Slawomir Solecki
(University of Illinois at Urbana-Champaign, USA)

Venue and Location: Lake Plaza Rotorua Hotel, 1000 Eruera Street, Rotorua, New Zealand.

Last date for the submission of abstracts: May 31, 2006.

Contact details:

Dr Jiling Cao and Dr Warren Moors
Department of Mathematics
The University of Auckland
Auckland, New Zealand
E-mail: `cao@math.auckland.ac.nz`,
`moors@math.auckland.ac.nz`

Ramanujan Mathematical Society

3–8th July 2006; 3–5th July 2006
Workshop and 6–8th July 2006 Conference

Venue

University of Hyderabad, Hyderabad

Details of Workshop

Workshop-main theme: Number Theory

1. 21st Annual Conference of Ramanujan Mathematical Society

Sl. No.	Speakers	Topic
1.	Dr. C. S. Dalawat, H.R.I., Allahabad	Basic Number theory
2.	Prof. Dipendra Prasad, T.I.F.R., Mumbai	Modular Forms

3.	Prof. R. Tandon, Univ. of Hyderabad	Hasse Minkowski Theory
4.	Dr. Suryaramanna, H.R.I., Allahabad	Introductory Analytic Number Theory

The workshop is meant for research scholars, exceptionally talented final year M.Sc. students, college lecturers and other interested academics.

Conference 5–8th July 2006

Invited Talks: Besides the RMS President's technical address and a special lecture entitled "Ramanujan Lecture" by an eminent mathematician, the following endowment lectures will be held:

1. Abdi Memorial Lecture
2. C. S. Venkataraman Memorial Lecture
3. J. N. Kapoor Memorial Lecture
4. M. N. Gopalan Endowment Lectures (2)

Invited Speakers are as Follows:

1. Prof. M. S. Narasimhan, T.I.F.R. Centre, Bangalore
2. Prof. Roddam Narasimha, IISc, Bangalore
3. Prof. V. Borkar, T.I.F.R., Mumbai
4. Prof. V. Srinivas, T.I.F.R., Mumbai (Lecture on latest Fieldmedallists)
5. Prof. I. B. S. Passi, Panjab University, Chandigarh
6. Prof. Vijay Chandru, IISc, Bangalore
7. Prof. Shiva Shankar, C.M.I., Chennai
8. Prof. L. V. Ramana, I.I.M., Indore
9. Prof. A. K. Pani, I.I.T., Mumbai
10. Prof. Mahendra Agrawal, I.I.T., Kanpur
11. Prof. R. Sridharan, C.M.I., Chennai

Paper Presentation: Those who want to present papers should send an abstract of the paper along with a hard copy of the paper so as to reach the **Local Secretary, 21st Annual Conference of Ramanujan Mathematical Society, University of Hyderabad, Hyderabad Pincode 500 046 on or before 31st May 2006**.

Prof. M. Venkataraman Best Paper Presentation Award

This award was instituted by Prof. M. Rajagopalan, Tennessee State University, USA in memory of his brother Prof. M. Venkataraman. The award carries a cash prize of Rs. 1000 and a citation and is presented to the young mathematician below the age of 35 who presents the best paper at each annual conference of the Society. Those who want to compete for this award should send two copies of the paper along with an abstract to the Local Secretary, RMS Conference, Department of mathematics, University of Hyderabad 500 046.

Registration: Those who want to attend the workshop/conference may apply to the Local Secretary at the address given above on a plain sheet of paper giving the following details:

Name, Address, Designation, Current Institution to which you are affiliated, Academic Record (for workshop only), Published Papers (for conference only). Registration Fee, D. D. No., Amount. Whether you want to present a paper or not.

Registration Fee for workshop: Rs. 50/-
Registration Fee for the conference will be Rs. 600/-

Those who want to attend both the conference and workshop need to give a registration fee of Rs. 600/- only which may be send through a D.D. drawn in favor of the Local Secretary, RMS Conference and should be sent to the Local Secretary so as to reach him on or before 31st May, 2006.

For Work shop: About 35 participants will be selected on all India basis. Selected participants for the workshop will be provided TA (II Sleeper train fare) and local hospitality.

2. Election Notice of Office Bearers of RMS

The term of office of present Executive Committee ends on 31st March 2006. The EC has nominated the following persons for the various positions as follows:

1. **President:** Prof. M. S. Narasimhan, Bangalore
2. **Vice-President:** Prof. Roddam Narasimha, Bangalore

3. **Secretary:** Prof. Alladi Sitaram ISI, Bangalore
4. **Academic Secretary:** Prof. Rajendra Bhatia, ISI, Delhi
5. **Treasurer:** Prof. K. N. Raghavan, IMSc, Chennai
6. **Librarian:** Prof. V. Balaji, CMI. Chennai.

Members:

1. Prof. Hans Gill, Punjab University, Chandigarh
2. Dr. Ambat Vijay Kumar, Cochin University of Science and Technology, Cochin
3. Prof. R. Tandon, University of Hyderabad, Hyderabad

The following are the Ex-officio Members of the EC.:

1. Prof. V. Kumar Murty, Editor in Chief JRMS, University of Toronto, Canada
2. Prof. Ravi S. Kulkarni, Editor in Chief RMS Lecture Notes Series in Mathematics, H.R.I., Allahabad
3. Prof. Phoolan Prasad, Editor-in-Chief, RMS Series on Little Mathematical Treasures, IISc., Bangalore
4. Prof. R. Parimala, Immediate Past President of RMS, T.I.F.R., Mumbai

Special Members:

1. Prof. R. Balakrishnan, Sastra, Kumbakonam
2. Prof. E. Sampathkumar, University of Mysore, Mysore.

Any life member of RMS can propose any other life member for any of the above positions. Such nominations if any should be recommended by two life members and send to Prof. A. Vijay Kumar, Secretary, RMS, Department of Mathematics, Anna University, Chennai on or before 20th February 2006. If no such nominations are received on or before this date it is presumed that the above persons nominated by the EC are elected to the various positions mentioned against their names.

3. Good News

Prof. M. S. Narasimhan has received the King Faisal International Prize in Science **Prof. R. Parimala** has received Third World Academy of Sciences Prize for Mathematics. Hearty Congratulations to both of them from RMS.

4. RMS International Conferences During 2006

1. INDIA–UK conference in Number Theory September 18–23, 2006. To be held in Institute of Mathematical Sciences, Chennai.

   Organising Committee:

   R. Balasubramaniam (IMSc)
   J. Coates (Cambridge)
   R. Heath-Brown (Oxford)
   R. Parimala (T.I.F.R.)
   D. Prasad (T.I.F.R.)
   A. Scholl (Cambridge)
   R. Sujatha (T.I.F.R.) and
   M. Taylor (Manchester)

   Sponsored by the Ramanujan Mathematical Society. The conference might also be partly sponsored by the London Mathematical Society. Further details will be posted in RMS website.
2. RMS International Conference in Discrete Mathematics will be held in IISc., Bangalore during 15–18th December 2006. For details see the RMS website mentioned above.

5. General Information

The contents of each issue of JRMS and the abstracts of papers as well as each issue of Mathematics News Letter are posted in the RMS website mentioned above.

6. RMS Annual General Body Meeting Notice

The annual general body meeting of the RMS will be held on 7th July, 2006 at 5.30 p.m. in Department of Mathematics, University of Hyderabad, Hyderabad. All members are requested to attend the same.

Agenda:

1. Welcome by the Secretary.
2. Announcement of the results of the election of new EC members by the Secretary.
3. Report of the Secretary for the year 2005–06.
4. Presentation of the minutes of the last annual general body meeting held at University of Calicut, Kerala during 2005.

5. Presentation of Audited Statements of Accounts for the year 2005–06 by the Treasurer.
6. Report of the Academic Secretary.
7. Report on the RMS International Conferences to be held during 2006.
8. Report on the following publications of RMS by the concerned Editors:

   (i) JRMS
   (ii) RMS Lecture Notes Series in Mathematics
   (iii) RMS Series on Little Mathematical Treasures
   (iv) Mathematics Newsletter

9. Venue for RMS annual conference during 2007.
10. Remarks by the President.
11. Any other matter with the permission of the Chair.
12. All members are requested to inform the change in address if any and their E-mail id's to Prof. E. Sampathkumar. E-mail: esampathkumar@eth.net.

Date: 17-01-2006

Dr. A. Vijayakumar,
Secretary, RMS,
Department of Mathematics,
Anna University,
Chennai.

King Faisal Prize for M. S. Narasimhan

The Ramanujan Mathematical Society is very happy to inform its members that its new president Prof. M. S. Narasimhan was chosen for the King Faisal Prize in Science for the year 2005. We invited Prof. T. R. Ramadas to write a brief summary of the work of Prof. Narasimhan. We are thankful to him for providing the following article.

The award of the King Faisal Prize for 2005 (in Science) to M. S. Narasimhan, jointly with Simon Donaldson, is a good occasion to take stock of his remarkable mathematical achievements and considerable contributions to the mathematical community in India and elsewhere.

M. S. Narasimhan was born on 7 June 1932, at Thandarai, in a rural part of what is now Tamil Nadu. His undergraduate years were spent at the Loyola College in Madras, where he was fortunate to be taught by Father Racine, himself a student of Elie Cartan and Hadamard. From Madras he went to Bombay as one of the first students to join the School of Mathematics at TIFR, then recently established under the leadership of K. Chandrasekharan. Here he learnt mathematics from the lectures of the likes of Laurent Schwartz and Warren Ambrose. His early work was in global analysis; a close study of the work of Kodaira-Spencer on deformation theory (during a long convalescence in Paris) then formed his taste in geometry.

M. S. Narasimhan's work is extraordinary both in its range and impact. In a career spanning 45 years, he has made fundamental contributions to algebraic geometry, differential geometry, the representation theory of semi-simple Lie groups and partial differential equations.

In 1965, Narasimhan and C. S. Seshadri proved that "stable vector bundles", a class of mathematical objects belonging to algebraic geometry, were the same as "flat (irreducible) unitary vector bundles", which are differential geometric objects. This link between algebro-geometric and transcendental objects has proved to be enormously significant. The Theorem has been generalized in many directions by Mehta–Seshadri, Donaldson, Uhlenbeck–Yau, Beilinson–Deligne, Hitchin, C. Simpson and others.

The moduli spaces of (semistable) vector bundles on curves, which Narasimhan and S. Ramanan studied extensively (along the way, proving a 50 year old conjecture in classical algebraic geometry), have proved to be of great interest from several different points of view, not least theoretical physics.

In work with G. Harder, Narasimhan used the Weil conjectures (which had then been recently proved by Deligne) to compute topological invariants of these spaces. Their computations have become the archetypes for several similar investigations of great importance. The Harder–Narasimhan filtration, which they introduced in this context, has been immensely useful for studying torsion-free sheaves on smooth varieties and in some questions in Number Theory. The work of L. Lafforgue, for example, uses an analogue of this filtration.

In differential geometry, Narasimhan and Ramanan proved the basic theorem about the existence of "universal connections". This result has been extensively used, for instance in the theory of Chern–Simons invariants and more recently in the work of Quillen on super-connections.

Together with K. Okamoto, Narasimhan made the first breakthrough in the proof of a conjecture of Langlands (on the concrete realization of Harishchandra's "discrete series representations"). He conjectured-and this was proved by his student R. Parthasarathy – that these representations would be realized on the space of s quare integrable solutions of Dirac operators on G/K. Yet again, this proved to be a very basic insight.

The fundamental theorem with T. Kotake, which vastly extended the well-known theorem on analyticity of solutions of a linear elliptic partial differential operator with analytic coefficients, further highlights the range and depth of Narasimhan's interests. This result has also been generalized in different directions. Kotake and Narasimhan also proved that fractional powers of a linear elliptic operator are (in the present-day language) pseudo-differential operators. This result was used in the original proof of the Atiyah–Bott fixed point theorem.

In the nineties Narasimhan's interests turned to mathematical questions arising from physics, more precisely from gauge theory and conformal field theory.

Narasimhan played a crucial role in the formation and development of schools in Algebraic Geometry, Differential Geometry and Lie groups at the Tata Institute. Former research students include S. Ramanan, M. S. Raghunathan, V. K. Patodi, R. Parthasarathy, T. R. Ramadas and N. Nitsure. He was also instrumental in the formation of the National Board of Higher mathematics, of which he served as the first Chairman.

Later, at Trieste, he was Director of the Mathematics Section at the Abdus Salam International Centre for Theoretical Physics, where visiting students and scholars from all over the world – from China, Latin America, the Middle East, and not least India – found in him a generous and incisive mathematical adviser and mentor.

Many honours have come his way-among them the Bhatnagar Prize, the Srinivasa Ramanujan Medal of INSA, the Padma Bhushan, and now the King Faisal Prize.

Narasimhan is very active; witness his steady output of high-quality publications. He is now back in India where, as Honorary Fellow of the Tata Institute, he continues to run seminars and be involved in mathematical policy initiatives.

First Announcement International Conference on Discrete Mathematics (ICDM 2006)

Organized by

Indian Institute of Science (IISc), Bangalore in association with Ramanujan Mathematical Society (RMS), India

Venue

Indian Institute of Science, Bangalore, India

Dates

December 15–18, 2006

The Ramanujan Mathematical Society was founded in 1985 with the ostensible purpose of promoting mathematics at all levels. During the last span of 20 years, RMS has witnessed a phenomenal growth. It has been publishing the Journal of the Ramanujan Mathematical Society since 1986. It also publishes the Mathematics Newsletter, a quarterly, that caters to the needs of the students and staff of colleges and other institutions in India. Recently, the Society has launched the publication of a Lecture Notes Series with financial support from the Government of India. It also publishes 'Little Mathematical Treasures', a series of books aimed to cater to the needs of high school and college students.

To mark its completion of 10 years, the Society organized in 1996 an International conference in Discrete Mathematics and Number Theory at Tiruchirapalli in India. The proceedings of the Discrete Mathematics part was published as a special number of the journal: DISCRETE MATHEMATICS (Vol. 206, 1999) and the Number Theory part was published in the CONTEMPORARY MATHEMATICS Series (# 210) of the American Mathematical Society.

To mark its 20 year completion, RMS now proposes to organize several conferences and one of them is the present international conference. This conference is being organized in association with the departments of Mathematics `(www.math.iisc.ernet.in)` and Computer Science and Automation `(www.csa.iisc.ernet.in)` of IISc.

The aim of the conference is to bring together leading researchers in discrete mathematics and thereby provide an opportunity to the vast body of students and young research scholars and teachers working in India in discrete mathematics and computer science to listen to the lectures and interact with them. In addition, it is expected that the Proceedings of the conference containing lectures of several leading mathematicians to serve as a good source book for researchers in several areas of discrete mathematics and applications.

Progamme Committee

S. Armugam
AKCE, Krishnan Koil, Tamil Nadu
s_armugam_akce@yahoo.com

Asha Rao
Royal Melbourne Institute of Technology, Australia
(asha@rmit.edu.au)

R. Balakrishnan (Co-Chair)
Srinivasa Ramanujan Centre, Kumbakonam, Tamil Nadu
mathbala@satyam.net.in

Renu Laskar
Clemsom University, Clemson, Atlanta, USA
relsk@clemson.edu

P. Paulraja
Annamalai University, Tamil Nadu
pprajaau@sify.com

Jaikumar Radhakrishnan
Tata Institute of Fundamental Research, Mumbai
jaikumar@tcs.tifr.res.in

Andre Raspaud
University of Bordeaux, France
raspaud@labri.fr

E. Sampath Kumar
University of Mysore
esampathkumar@eth.net

C. E. Veni Madhavan (Co-Chair)
Indian Institute of Science, Bangalore
cevm@csa.iisc.ernet.in

Venkatesh Raman
Institute of Mathematical Sciences, Chennai
vraman@imsc.ernet.in

A. Vijayakumar
Cochin University of Science and Technology, Kochi
ambatvijay@rediffmail.com

The conference will consist of plenary talks, invited lectures, mini-symposia and poster sessions, encompassing all areas of discrete mathematics and its applications.

A mini-symposium will involve three speakers, each delivering a talk for 45 minutes, on a single chosen theme.

The Topics for the Mini-Symposia Include:

Domination theory
Graph colouring
Spectral graph theory
Graph decompositions
Design theory
Coding theory and cryptography
Approximation and randomized algorithms
Combinatorial optimization
Discrete mathematics in other sciences
Proposals for conduct of mini-symposia in other topics are also being planned.

All poster session papers will be reviewed.

Current List of Speakers:
Plenary Speakers:
J. Nesetril, Prague
Carsten Thomassen, Denmark

Invited Speakers:

R. B. Bapat, Indian Statistical Institute, New Delhi
S. A. Choudum, Indian Institute of Technology, Chennai
Diana Donovan, University of Queensland, Australia
S. B. Rao, Indian Statistical Institue, Kolkata
K. Thulasiraman, Okalahoma State University, USA
Xuding Zhu, National Sun Yat-Sen University, Taiwan

The Following Experts have Agreed to Coordinate Mini-Symposia in the Topics Noted Against their Names:

S. Armugam	Domination Theory
Asha Rao	Coding Theory

P. Paulraja	Graph Decompositions
Andre Raspaud	Graph Coloring
H. B. Walikar	Spectral Graph Theory

Other plenary and invited speakers are being contacted.

Conference Proceedings:

The conference proceedings will contain all the plenary, invited and mini-symposia talks. The proceedings will be published in the Lecture Notes Series of the Ramanujan Mathematical Society.

The proceedings will be distributed to all the participants along with the other conference materials on the first day of the conference.

Registration Fees:

Participants from Outside India (Including Invited Speakers)

US $200 per person
For each accompanying person US $150.

This registration fee entails accommodation and food in the guest houses in and around IISc and also the conference kit and a copy of the proceedings.

Participants from within India: Rs. 1000/- per person.

This registration fee entails breakfast, lunch, conference kit and a copy of the proceedings. Accommodation will be arranged on request at nominal rates. If adequate grants are forthcoming, TA and DA will be paid to the participants from within India (However, we are unable to commit ourselves for this at this point of time).

Deadlines:

Talks by invited and plenary speakers/ Lecture notes of mini-symposia	: July 31, 2006
For submission of papers for poster sessions	: July 31, 2006
Acceptance notification of poster papers	: September 15, 2006
Final receipt of poster papers	: October 15, 2006
For registration	: August 31, 2006

Details of payment of registration fees, travel, tourism information on Bangalore, nearby Mysore and other places will be posted in the official websites of the conference:

1. The RMS website : `www.ramanujanmathsociety.org`
2. IISc website : `www.math.iisc.ernet.in/imi` (the link to IISc Mathematics Initiative page)

Bhargava–Clay Fellowship of the RMS

The Ramanujan Mathematical Society invites nominations for the first Bhargava–Clay Fellowship of the RMS.

The Fellowship will support one or perhaps two outstanding graduate students in India who are in the advanced stages of their Ph. D. work in any area of pure or applied mathematics. The primary selection criteria for the Fellowship are the exceptional quality of the candidate's research and the candidate's promise to become a mathematical leader.

The award will consist of scholarship support for one year of Rs. 50,000.

Nominations consist of a nominating letter from a faculty member describing the candidate's current research and future potential, a curriculum vita, and any relevant supporting material. Nominations should be sent preferably by E-mail, to `narasim@math.tifrbng.res.in` with a hard copy posted later to the following address:

Professor M. S. Narasimhan,
T.I.F.R. Centre,
Indian Institute of Science Campus,
Bangalore 560 012, India.

The deadline for nominations for this year is May 15, 2006.

Acknowledgements: These newly instituted Bhargava–Clay Fellowships of the RMS as well as travel grants for the Annual Meeting of the RMS are available as part of the Clay Mathematics Institute Research Prize to Professor Manjul Bhargava; we are grateful to both CMI and Prof. Bhargava for their generosity in making this possible.

Book Review
Partition Theory

Authors: A. K. Agarwal, Padmavathamma and M. V. Subba rao

Publisher: Atmaram and Sons, Chandigarh, India, 2005
Pages x + 307, Price Rs. 595

Review: Within the larger realm of number theory, the theory of partitions occupies a somewhat neglected corner. Many books on number theory mention partitions only briefly, if at all. Yet research on partitions is an ongoing activity. Most researchers in this area obtain their results using modular forms, combinatorial methods or elementary methods.

Until recently, the only monograph devoted entirely to partitions was "The Theory of Partitions" by George Andrews, which first appeared in 1974. It is therefore entirely appropriate that a new book devoted to partitions has been published, entitled "Partition Theory". The three authors, A. K. Agarwal, Padmavatthama and M. V. Subba rao, have all done extensive research on partitions. In addition, all three are natives of India, although the third author has resided for some time in Canada.

The topics covered in this 307 - page book include fundamentals of partitions, n-color partitions, 4-way partition identities, partitions with d(a) copies of a (where $d(n)$ denotes the number of divisors of n), generalized Frobenius partitions, product partitions and parity of partition functions.

In certain parts, the writing style of the book is somewhat terse, following the model of definition, theorem, example, proof. Some elementary results are simply quoted and not proved, although the reader is referred to other sources for the proofs.

According to the authors, this book is intended for "senior graduate and research students", as well as those who are already experts in partition theory. The book contains quite numerous references to research articles, many by the authors themselves.

In summary, "Partition Theory" is a welcome addition to the literature that concerns this fascinating subject.

Reviewed by:

Dr. Neville Robbins
San Francisco State University, U.S.A
E-mail: robbins@math.sfsu.edu

Report on the International Conference on Geometric Function Theory, Special Functions and Applications

January 02–05, 2006, Pondicherry

Balantidiases Govt. College for Women, Pondicherry and Form d' Analysts, Chennai jointly organized an International Conference on Geometric Function Theory, Special Functions and Applications during January 02–05, 2006 at Pondicherry. The inaugural function of the conference was presided over by Thiru N. Rangasamy, Chief Minister, Government of Pondicherry. Prof. A. Gnanam inaugurated the conference and Prof. R. Balasubramanian, the Director of the Institute of Mathematical Sciences spoke on the aim and activities of the Form d' Analysts. The keynote address was delivered by Prof. R. W. Barnard, Texas Tech University, Lubbock, U.S.A.

The main theme of the conference was Geometric Function Theory with special reference to part played by Special Functions thereof. More than one hundred and fifty four delegates working in this field were gathered and of them thirty eight were from outside India. All of them had taken keen interest in the deliberations.

There were plenary lectures by senior eminent specialists in this field and thirty invited talks by young subject-exports. Papers were presented by promising upcoming research scholars working in this field. Many young Indian delegates were very much benefited by this meeting in which they can interact with experts from all over the world.

The conference was cosponsored by other institutions interested in this field such as M O P Vaishnav College, Rajiv Gandhi College of Engineering and Technology, Achariya Arts and Science College, and North Orissa University. The organizers are grateful to National Board for Higher Mathematics, Council for Scientific and Industrial Research, Department of Science and Technology, New Delhi and Department of Science and Technology, Pondicherry for the financial support extended for the conduct of the conference and particularly thankful to National Science Foundation, U.S.A. for deputing sixteen U.S. Mathematicians working in this field for this conference.

Mr. R. Narayana, I.A.S., state election commissioner, Pondicherry was the Chief Guest of the Valedictory function and distributed certificates to the participants.

The proceedings of the conference will be brought out soon as a special volume of the Journal of Analysis and will be edited by Prof. S. Ponnusamy, Academic Secretary of the conference and Prof. R. W. Barnard, National Science Foundation (USA) coordinator of this conference.

Convener

UGC Sponsored National Seminar on Number Theory and its Applications

Providence College for Women, Coonoor Nilgiris
Tamil Nadu

March 1st and 2nd, 2006

A Condolence Resolution: Prof. M. V. Subbarao had enriched the field of Number Theory by his continuous researches spanning over a period of six decades. His valuable contribution to Number Theory has been well recapnized in the national as well as international circles. He collaborated with famous Number Theorists like Paul Evdos, George E. Andrews and many others. He trained many young researchers in the field. He started his teaching career as a lecturer during 1950's in the Madras Govt. Educational service and then moved over to the newly formed Sri Venkateswara University Thirupathi during 1956. From Thirupathi, he went to USA in 1963 and finally settled down in the University of Alberta, Canada, where he adorned the professional chair with distinction, till his last breath on 15th Feb 2006 in Edmonton, Canada. The far reaching influence of Prof. M. V. Subbarao as his students in India and abroad is conspicuous in their continuing survival work in the respective research branches of Number Theory and in turn bequeathing this legacy to their students. Thus Prof. M. V. Subbarao will be remembered more as an institution rather than a highly accomplised individual. The chairperson, convenor, members of the organising committee and all the invitees and participants at the National Seminar condoled the denuise of Prof. M. V. Subbarao by observing silence for a couple of minutes in honour of the members of the departed soul on 1st march 2006 at 11 am. Subsequently this condolence resolution was passed during the valedictory session on 2nd march 2006 at 4.30 pm.

Call for Papers 1st International Conference on Nano-Networks (Nano-Net 2006)

September 14–16, 2006, Lausanne, Switzerland
Conference Web-site: www.nanonets.org

Sponsors: Create-Net, ICST, EU (IST-FET), IEEE-CAS and CEDA

Conference Scope: Nano-Net 2006 positions itself at the intersection of two worlds, namely, emerging nano-technologies on one-side, and network/communication theory on the other side. The conference will address the new communication paradigms that derive from the transition from micro- to nano-scale devices. The related degrees of freedom and constraints associated with these new technologies will change our notions about efficient network and system design. Nano-Net provides a multi-disciplinary forum for the discussion of new techniques in modeling, design, simulation, and fabrication of network and communication systems at the nano-scale.

The conference invites original technical papers that have not been previously published and are not currently under review for publication elsewhere. Contributions addressing all subjects pertaining to nano-technology and networking are solicited. Suggested topics include, but are not limited to, the following:

1. Networks-on-Chip (NoC)
   - Network architectures and topologies
   - NoC performance and trade-off analysis
   - Energy efficiency and power management
   - CAD flows for NoCs and MP-SoC platforms
   - Fault tolerance and reliability issues
2. Innovative System Interconnects
   - Nano-technologies and devices for on-chip interconnects (CNTs, semic., metallic and DNA-templated nanowires)
   - Molecular, optical and wireless interconnects
   - Interconnects for non-charge-based devices
3. Architectures and Systems for Nano-Networks
   - Implications of nano-networks on processor and memory architectures
   - New computing paradigms for nano-networks
   - Hybrid CMOS-nano network systems
   - Self-organization in nano-scale systems
4. Novel Information Theory Aspects of Nano-Networks
   - Re-configurability of nano-networks
   - Statistical mechanics approach to nano-communications
   - Routing/addressing issues in nano-networks
   - Nano-coding, Applications of complex networks theory
5. Modeling, and Simulation of Nano-Networks
   - Physical characterization/modeling of nano-scale interconnects and devices
   - Fault-tolerant aspects of nano-devices
   - Self-healing properties of nano-networks
   - "Nano-CAD"
   - Bio-inspired aspects
6. Nano-Scale Wireless Communications
   - Nanoscale Electromagnetics
   - Propagation models for nanoscale communications
   - Planning and optimization
   - Nanoscale antennas, nano-arrays
   - EMI aspects at nanoscale
   - Reconfigurability issues

Submission Instructions: Prospective authors are encouraged to submit a PDF version of the full paper in English language. Papers are limited to 5 two-column pages, in a font no smaller than 10-points. Style guides and further information are available on the conference website: `http://www.nanonets.org`. Presentation will be oral or in a poster format, as deemed more appropriate by the Technical Program Committee.

Publication: All submitted papers will be subject to a rigorous peer-review. Accepted papers will be published by the IEEE in the Nano-Net Conference Proceedings, and made available online through IEEE Xplore. A selected number of high-quality papers will be considered for publication in the ACM Journal on Emerging Technologies in Computing Systems (JETC). Also, selected papers with a strong nanoscale device component will be reviewed for inclusion in a special section of the IEEE Transactions on Nanotechnology.

Special Sessions: Proposals for Special Sessions, especially focusing on future applications of nano-networks are encouraged. Potential Special Session organizers should submit a proposal of at most 5 pages, including scope/motivation of the session, list of invited papers (still subject to peer-review) and short-bio of the organizer(s). Proposals should be submitted by E-mail to: `special-sessions@nanonets.org` (Prof. Mircea Stan).

Important Dates:

Special Sessions Proposals Due: April 1, 2006
Paper Submission Due: May 1, 2006
Notification of Acceptance: July 1, 2006
Final manuscript Due: August 1, 2006

Ramanujan Prize for Young Mathematicians from Developing Countries

The Abdus Salam International Centre for Theoretical Physics (ICTP) is pleased to invite nominations for the 2006 Ramanujan Prize for young mathematicians from developing countries. The Prize is funded by the Niels Henrik Abel Memorial Fund. The Prize carries a $10,000 cash award and an allowance to visit ICTP for a meeting where the Prize winner will be required to deliver a lecture.

The Prize will be awarded annually to a researcher from a developing country less than 45 years of age on 31 December of the year of the award, who has conducted outstanding research in a developing country. Researchers working in any branch of the mathematical sciences are eligible. The Prize will be awarded usually to one person, but may be shared equally among recipients who have contributed to the same body of work.

The Prize winner will be selected by ICTP through a committee of five eminent mathematicians appointed in conjunction with the International Mathematical Union (IMU). The deadline for receipt of nominations is July 31, 2006.

The first winner of the Prize for 2005 is Professor Marcelo A. Viana from IMPA, Brazil, and the award Ceremony was held at ICTP on December 15, 2005.

Please send nominations to `director@ictp.it` describing the work of the nominee in adequate detail. Two supporting letters should also be arranged.

Note: More about ICTP activities may be obtained from its web page at `http://www.ictp.it`

Sliver Jubilee Year of UGC-Special Assistance Programme

International Conference on Frontiers in Fluid Mechanics

October 26–28, 2006

Organized by

UGC-CAS in Fluid Mechanics
Department of Mathematics, Bangalore University
Bangalore 561 001, India

About the Conference: The UGC had recognized the Department of Mathematics, Bangalore University for DSA under SAP in the year 1980 with thrust area 'Fluid Mechanics', and this programme started functioning from 1981. Since then, the UGC has continued DSA Programme till 2002, and then upgraded it to a Center for Advanced Studies in Fluid Mechanics in 2002.

To commemorate this event, we intend to celebrate the silver jubilee of SAP in 2006 by organizing an International Conference on "Frontiers in Fluid Mechanics" by inviting the collaborators of the centre from India and abroad and also inviting distinguished researchers within and outside India. The scientists from USA, Canada, Germany, Hong Kong and Japan have collaborated with the Centre and they have agreed to attend this International Conference and to celebrate the silver jubilee of the SAP. Several collaborators within India are also participating in this unique event.

Scope of the Conference: The emergence of newer fields have played a prominent role in the advancement of Fluid Mechanics with potential applications in Science, Engineering and Technology. The following are the themes of the conference:

Modelling of nano, smart and chiral materials
Flow through and past porous media
Heat a nd mass transfer processes
Laser radiation, Inertial fusion technology and plasma
Environmental pollution
Biomedical and bioengineering problems
Laminar and turbulent flows
Waves and dispersion phenomena

The conference also provides ample opportunities to interact among the researchers working in fluid mechanics and allied fields.

Patrons : Vice Chancellor, Bangalore University
Advisor : Prof. N. Rudraiah
Convener: Prof. M. Venkatachalappa

International Conference on Frontiers in Fluid Mechanics

October 26–28, 2006

Registration Form

1. Name of the Participant:
2. Address for Correspondence:

   Fax: Ph.:
   E-mail:
3. Gender: Male/Female
4. Title of the Paper:
5. Accommodation: required/not required

Kindly send the registration form to

Prof. I. S. Shivakumara (Coordinator)
UGC-Center for Advanced Studies in Fluid Mechanics
Department of Mathematics, Bangalore University
Bangalore 560 001, India
E-mail: isshivakumara@hotmail.com

Abstract Submission: Those interested in presenting research papers should send an abstract of about 250 words by E-mail to:
iskumar2006@gmail.com or iskumar2006@yahoo.com or mvenkatachalappa@hotmail.com on or before July 15, 2006.

Financial Assistance: Conference funds are limited. However, partial travel grants and living expenses to a few deserving young participants may be provided. Such grant aspirants may apply along with recommendation letters through concerned supervisors and heads of departments to the Conference Convener.

There is no registration fee and reasonably comfortable accommodation will be arranged for selected participants.

Invited Lectures: There will be key note addresses, plenary and invited lectures from experts in the field of fluid mechanics. Besides there will be an oral and poster sessions.

Harish–Chandra Research Institute, Allahabad

Visiting Students' Study Programme (VSSP) in Mathematics

(5th–23rd June 2006)

Harish–Chandra Research Institute conducts the Visiting Students' Study Programme (VSSP) during summer every year. The aim of this programme is to introduce interesting mathematics to undergraduates who have spent at least three years at college and motivate them to take up a career in mathematics research. The programme involves intensive training on selected topics in mathematics for a period of three weeks.

Students who do very well at the VSSP are often considered for admission to the graduate program. The programme has, in the past, provided HRI with several graduate students.

The VSSP this year will be held from *5th to 23rd of June 2006*. Interested students can apply in the enclosed format, along with *two recommendation letters* from their present teachers to:

The co-ordinator, Math VSSP 2006
Harish–Chandra Research Institute
Chhatnag Road, Jhunsi
Allahabad 211019, Uttar Pradesh.

Preference will be given to M. Sc. I year and II year students, however highly motivated B. Sc. final year students are also encouraged to apply.

The deadline for receiving completed application form is **10th April 2006**. The list of selected students will be available in the VSSP web page http://www./mri.ernet.in/∼mathvsp from 15th April onwards and they will also be intimated by post by 30th April.

All the information about VSSP is available in the above web page. For any information regarding VSSP, that is not available in the web page, E-mail to mathvsp@mri.ernet.in

Selected students will be reimbursed sleeper class train fare (by shortest route) from their place of residence to Allahabad and back. They will also be provided free accommodation in HRI campus and a honorarium of Rs. 125/- per day during the programme days.

Ratnakumar P. K
Co-ordinator, Math VSSP 2006, HRI, Allahabad

Visiting Students' Study Programme 2006 in Mathematics

(5–23 June 2006, HRI Allahabad)

Application Form

(1) Name (in Block Letters):

(2) Address for Communication:

(3) E-mail address:

(4) Highest degree completed:

(5) Referees' name* (with designation and *E-mail address*):

i: *E-mail*:

ii: *E-mail*:

(6) Academic records ⋆ (10 +2 onwards):

Sl No.	Qualifing Exam	College or University	Year of Passing	Percentage of Marks	% of Marks in Mathematics
1					
2					
3					

Those who have not completed M. Sc. final year should give the details of M. Sc. first year marks.

Signature of the candidate:

Station:

Date:

⋆ Please attach copied of mark lists

* Applicant should send the recommendation letters in an envelope sealed and signed by referee, along with the application.

Announcement and Call for Papers Third National Conference on Applicable Mathematics

In Wave Mechanics and Vibrations (WMVC-2006)

October 15–17 (Sunday–Tuesday), 2006

Organised Jointly by

The Jaypee Institute of Engineering & Technology (JIET), Raghogarh 473 226 (Guna), M.P and
The Von Karman Society for Advanced Study and Research in Mathematical and Social Sciences,
Old Police Line, Jalpaiguri 735 101, West Bengal.

Scientific Committee:

Prof. N. J. Rao (Conference Director)
Dr. P. Biswas (Conference Co-Director)
Prof. V. K. Agarwal (Meerut/UP)
Prof. S. Karanjai (University of North Bengal)
Prof. Y. Nath (IIT, Delhi)
Dr. M. Ganapathi (IAT, Girinagar, Pune)
Prof. J. Raamachandran (IIT, Chennai)
Dr. Jyotirmay Mukhopadhyay (Principal), (Kolkata)
Prof. P. K. Datta (IIT, Kharagpur)
Dr. V. Balamurugan (CEAD, DRDO, Avadi, Chennai)
Prof. Bhola Ishwar (Muzaffarpur University)
Dr. Vipin Tyagi (JIET), Organising Secretary

Aims and Objectives of WMVC: Over the years during the 20th Century every branch of Mathematics, Pure or Applied, has been enriched and developed through continuous efforts, cultivation and contribution by researchers in different fields like Social, Physical, Mathematical and Engineering Sciences. Analysis of scientific thoughts and problems of science and engineering have been made through the language of Mathematics and as a result Mathematics as a subject has got continuous and intensive development through the passage of time.

The purpose of holding such conferences is to bring together, on a common platform, scientists, engineers and researchers along with other eminent academic personalities in selected pure and applied branches. They would get ample scope to make exchange of views, ideas and thoughts besides presentation of papers and development of Mathematics as results of intensive research in the area of the conference-themes.

Considering the importance of the titled conference in view of enormous applications of the topics in wave mechanics and vibration sciences, structural mechanics, aeronautics, nuclear and atomic power reactors, biomedical engineering and in different branches of engineering, holding of such a conference of national and global importance will open scope for attending by those persons who have been working in the area of the conference topics. It is expected that different mathematical methods and innovative skills employed in different class of problems of mathematical sciences and possible future developments will be intensively discussed during the conference.

Conference Sub-topics:

- Nonlinear Phenomena in Mathematical and Physical Sciences
- Mathematical and Computational Methods, Astrophysical Problems
- Mechanics of Solids and Structural Mechanics
- Elastic Wave Mechanics, Fluid Mechanics and Plasma Physics

- Vibrations of Beams/Plates/Shells including Thermal and Random Vibrations
- Thermal Stresses, Thermal Buckling and Post-Buckling Analysis
- Mechanical Behaviour of Structures at Elevated and Criogenic Temperatures
- Bio-medical Engineering
- Seismic Analysis
- Boundary-value Problems Related to Engineering Sciences
- Recent Developments in Applied Mathematics and Applications.

Location, Climate and Places of Tourist Interest: The **Venue of the Conference** will be the Conference Hall of the JIET which is at Raghogarh on the Agra-Bhopal-Jaipur Road (NH-3) and connected by the Kota-Bina/Gwalior-Guna Railway Sections. Also well-connected by Road from Gwalior (245 Kms), Bhopal (190 Kms) and Indore (256 Kms) and served by the MPSTC The Climate of Raghogarh in mid-October is generally warm and pleasant with little chance of rainfall. There are many nearby important places of tourist and religious importance. Services of Local Travel Agents are available to visit such places. For more details of location, please visit/consult the Website: `jiit.ac.in/jiet.ac.in`

Abstracts:

* Two copies of Abstracts (150–200 words)	Before May 31, 2006
* Notification of Acceptance/Invitation	Before July 15, 2006
* One copy of the 6-page manuscript of paper with CD neatly typed in MS-Words and with intimation/confirmation for participation with at least 50% Delegate Fee	Before August 15, 2006

Technical Sessions: Technical Sessions are planned for Invited Speakers (Thirty minutes each) and Contributed Papers (Twenty minutes each). **OHP will be provided for presentation of papers**. Authors may bring personal Laptops for Power-Point Presentation of Papers.

Registration Fees: Nominal Registration Fee is payable by all the Authors, Co-Authors and Accompanying Persons as prescribed below.

Sponsored/Institutional Delegates	-Rs. 750/- (Confirmation requested before August 15, 2006)
Author/Co-Author (Per Person)	-Rs. 600/- (Before August 15, 2006)
Accompanying Persons	-Rs. 500/- (Before August 15, 2006)
Late/Spot Registration Fee	-Rs. 750/- (Authors/Co-Authors)

The above Fees are to be paid by DD (or on-line through SBI having Core-Banking Facility) in favour of "Von Karman Society for Advanced Study and Research in Mathematical and Social Sciences", A/C No. 10176348340 at State Bank of India, Jalpaiguri Town Branch, Jalpaiguri 735 101 (WB), (SBI Code No. 2070) followed by due intimation of money transfer if made through core-banking facility.

Special Offer: For organisational convenience intending delegates are advised to co-operate by way of sending their early registration fee (before August 15, 2006) which will ensure cost-free simple accommodation (in GH/Student's Hostel "on first come first deserve basis " until limited cost-free accommodation is exhausted) plus local hospitality for them, otherwise it may not be possible to provide accommodation free of charge for late registered delegates.

Financial Support: Organising committee is unable to provide any travel support. All delegates are advised to obtain their travel support from their own institutional sources. College and University Teachers are advised to obtain travel support from the UGC's PTAC Grant.

Conference Publications: Book of Abstracts of Papers will be available during the conference. Publication of the Volume-A of the Proceedings of the Conference may be possible before the conference depending on submission of one hard-copy of the Full-length Manuscript of the paper (with CD and neatly typed in MS-WORDS) before July 15, 2006. Otherwise the Proceedings will be published after the Conference.

Abstract of Papers and Any Communication Should Be Sent To the Following Addresses:

Dr. P. Biswas (Conference Co-Director)
Head, Vibration Research Group and Executive Secretary,

Von Karman Society for Advanced Study and Research in Mathematical and Social Sciences,
Old Police Line, Jalpaiguri 735 101, West Bengal
Phone: 03561-256894
E-mail: biswas_paritosh@yahoo.com

For Further Information (and also on how to reach JIET) Please Write To

Dr. Vipin Tyagi, Organising Secretary
Department of Mathematics,
Jaypee Institute of Engineering and Technology (JIET)
Agra–Bhopal Road, Raghogarh 473 226, Dist-GUNA (Madhya Pradesh)
E-mail: tyagivipindr@rediffmail.com
or
vipintyagi@jiet.ac.in

Advanced Training in Mathematics Schools (Supported by National Board for Higher Mathematics)

An Advanced Instructional School in Functional and Harmonic Analysis

3–28 July, 2006, I. S. I. Bangalore

Conveners

E. K. Narayanan, T. S. S. R. K. Rao

National Coordinating Committee:

Director : R. S. Kulkarni, HRI, Allahabad
Secretary : J. K. Verna, IIT, Bombay
Members : S. D. Adhikari, HRI, Allahabad
S. Deo, HRI, Allahabad
S. Madan, IIT, Kanpur
I. B. S. Passi, HRI, Allahabad
R. A. Rao, TIFR, Mumbai

Advanced Training in Mathematics Schools (ATM Schools) are a joint effort of more than 50 active researchers across the country with support from the National Board for Higher Mathematics. The objective of these schools is to impart basic knowledge in algebra, analysis and topology in the Annual Foundation School (AFS) series and advanced knowledge in diverse areas of mathematics in the Advanced Instructional School (AIS) series.

Presently we invite applications for participation in AIS in Functional and Harmonic Analysis.

Eligibility for Participation: Applications are invited from doctoral students in mathematics who have qualified in the UGC/CSIR JRF/SRF examination, or have won the National NBHM Ph. D. awards or who are getting scholarships in other Institutes. M. Sc. students with exceptional background supported by recommendations from their teachers may be considered. (See the website for more details). In addition, a few post-doctoral fellows, young teachers at universities/colleges and Ph. D. students in other areas, will be considered.

Financial Support: Selected participants will be paid III-AC return train fare from their place of work/home town to the venue as per NBHM norms and provided with accommodation and local hospitality.

How to Apply: Application forms may be downloaded from the website: http://www.math.iisc.ernet.in/~atm
or
http://www.math.iitb.ac.in/atm

The syllabus and the other information about AIS is available on these websites.

Applications May Also Be Made on Plain Paper, Giving the Following Information In the Same Order:

Name, Date of Birth, Gender, Institute/Department, Scholarship, Areas of interest, ATM/MTTS Schools attended, Address for correspondence, City, State, Pincode, Names and E-mail addresses of two teachers who are writing recommendation letters for you, Academic Record: B. Sc./M. sc./ with names of the Institutes. This should be attested by head of the institute. (Do not send any copies of certificates.)

Mail the Completed Forms To:

Prof. E. K. Narayanan
Dept. of Mathematics,
Indian Institute of Science
Bangalore 12
Phone: 080-22933270, 22932711
Fax: 80-2360 0146

so as to reach before **20 April 2006.**

Contact E-mail: `atm@math.iisc.ernet.in`

List of selected candidates will be posted on the websites on 24th April 2006.

Probable Resource Persons:

E. K. Narayanan	T. S. S. R. K. Rao
S. Thangavelu	Bhaskar Bagchi
Alladi Sitaram	Gadadhar Misra
V. Muruganandam	

You may download the Mathematics Newsletter from the RMS website at
`www.ramanujanmathsociety.org`