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Geometric reductivity – A quotient space approach

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1. Introduction

Mumford's *Geometric Invariant Theory* or *GIT* is a major technique for finding quotients of algebraic schemes acted upon by reductive algebraic groups. It has been successful in finding solutions to moduli problems in the category of *algebraic schemes*. In the first edition (i.e., the 1965 edition) of *Geometric Invariant Theory* [13], Mumford restricted himself to algebraic schemes over fields of characteristic zero. In order to make his theory applicable over fields of arbitrary characteristic, he made the following conjecture in the Preface to the first edition of *Ibid.* (a conjecture subsequently proved by Haboush [5] in 1975):

Let G be a reductive algebraic group over an algebraically closed field k. Then G is GEOMETRICALLY REDUCTIVE*, i.e., for every finite-dimensional rational G-module V and a G-invariant point $v \in V$, $v \neq 0$, there is a G-invariant homogeneous polynomial F on V of positive degree such that $F(v) \neq 0$.*

As a consequence, it can be shown that if $X = \operatorname{Spec} A$ is an algebraic scheme on which our reductive algebraic group G acts, then the affine scheme $Y = \operatorname{Spec} A^G$ is an algebraic scheme, i.e, the ring of invariants A^G is finitely generated as a k-algebra (a result of Nagata [14]) and the canonical morphism $f \colon X \to Y$ (induced by the injection $A^G \hookrightarrow A$) is surjective. Further, if Z is a closed G-stable subset of X, then $f(Z)$ is closed in Y, and if f separates *disjoint closed G-stable* subsets of X, i.e., given two disjoint closed G-stable subsets Z_1 and Z_2 of X, then $f(Z_1)$ and $f(Z_2)$ are also disjoint. In other words, $f \colon X \to Y$ is what is called a *good quotient* [18]. These results are in fact *equivalent* to the conjecture.