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The Life and Works of K. G. Ramanathan

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Abstract. Prof. K. G. Ramanathan was a legendary Indian Mathematician, working in Number Theory and a prolific Institution builder. Apart from this, he was an excellent teacher and influenced several brilliant students. In this article, we overview his life and discuss some of his significant mathematical contributions.

Keywords. Ramanujan, Modular forms, Mock modular forms.

2020 Mathematics Subject Classification: Primary 01A70, 11A07 11A25; Secondary 01A75, 01A73, 01A72.

1. Early life and education

Figure 1. K. G. Ramanathan (1920–1992)

Kollagunta Gopalaiyer Ramanathan, or popularly known to his colleagues, friends and students as KGR, was a distinguished Indian mathematician, well known for his outstanding work in number theory and contributions in building a strong school of mathematics research and teaching in India. His influence on the mathematical scene in India post-independence was huge. Ramanathan was born on the 13th of November, 1920 in Hyderabad, a city which is situated in southern India, and currently the capital of Telangana. His mother, Ananthalaxmi, died at an early age. His father's name was Kollagunta Gopal Iyer. Apart from him, there were two sisters and one brother in his family. Ramanathan's schooling was done from Wesleyan Mission High School, Secunderabad. He obtained his undergraduate degree from Osmania University in Hyderabad in 1940 and his Master's Degree from the University of Madras in 1942, studying at the Loyola College. He also worked as a Research Scholar and taught for a while in the latter before moving to the United States for higher studies. During his stay at Madras, Ramanathan came in contact with many famous mathematicians like Professors Vaidyanathaswamy and Vijayaraghavan, who inspired him. His first research paper was on Demlo Numbers [1], and it was published in 1941. During this period, he published some other papers as well, in reputed journals like the Mathematics Student and Journal of the Indian Mathematical Society.

In 1951, he received his Doctoral Degree in Mathematics from Princeton University, where his advisor was none other than **Emil Artin**, considered as one of the co founders of the theory of modern abstract algebra (along with Emmy Noether). The title of his doctoral thesis was "The Theory of Units of Quadratic and Hermitian Forms." At Princeton, he came in contact with legendary mathematicians like Hermann Weyl and Carl Siegel, whose mathematics heavily influenced his own research later. When he was the Institute for Advanced Study, Ramanathan assisted Prof. Weyl, one of the most significant Mathematicians and Theoretical Physicists of the last century, who was a Member of the Institute at that time. The mathematics he learnt during these crucial early years had a lasting effect on his entire career.

2. Returning back to India

After completing his PhD in the USA, Ramanathan returned to India in 1951 and joined K. Chandrasekharan at the School of Mathematics, Tata Institute of Fundamental Research (TIFR), Mumbai. The department had just started its journey. The decade of 1950s is very crucial for Indian mathematics, as it was during that time when Ramanathan and Chandrasekharan, with full support and encouragement from Homi Jehangir Bhabha, FRS (the first Director of the Institute), were working on transferring the department

Figure 2. Part of the group photo at the 1955 Tokyo Nikko Conference. **Top** Row, from left – Morikawa, Serre, Taniyama Yamazaki; **Third** Row, from left - Tsuzuku, Takashi Ken Ono, Azumaya, Nagai, Keiko Satake, Ichiko Satake, Hayashida; **Second** Row, from left – Shimura, Shinziro Mori, Hitotumatu, Terada, Nakano, Kawai, Ishii; **Seated**, from left – Iyanaga, Chevalley, Brauer, Zelinsky, Weil, KGR, Iwasawa. (taken from Photo Key in [23])
Courtesy - **My Parents' Generation**, Ken Ono and Amir D. Aczel [27].

into a notable centre for mathematics. They attracted some amazing research scholars and also invited many leading mathematicians like C. L. Siegal, Samuel Eilenberg, Laurent Schwartz and Kiyosi Ito to visit the Department and deliver courses in many areas of advanced mathematics. For example, Eilenberg gave a course on the recent developments in Algebraic Topology in 1955, and Schwartz lectured on complex analytic manifolds. [19] The work culture in those days was such that the students were given complete academic freedom to read and explore whatever they wanted, as long as Ramanathan and Chandrasekharan themselves were convinced that the student was serious enough. They however, asked difficult questions as Interview panel members during the time of admission of PhD students. With his unending enthusiasm for good mathematics and dedication towards the development of teaching and research of math in our country, Ramanathan was successful in setting up a strong Number theory school at TIFR. He had amazing international connections, by attending various International Conferences and Meetings. For example, he attended the landmark and immensely successful **Tokyo Nikko Conference** on Algebraic Number Theory in September 1955 (where his advisor Artin was also present), and gave a talk titled "Units of fixed points in involutorial algebras" [23]. The event was the first major Mathematics event held in Japan after the Second World War, and is special for many reasons. It attracted 77 participants, which included legendary number theorists from the West, like Artin, Chevalley, Deuring, Serre, Weil, as well as the new generation of Japanese mathematicians including Iwasawa, Satake, Shimura, Taniyama, Takashi Ono (father of the famous mathematician Ken Ono). At an after dinner impromptu lecture (which was not a part of the official program), Weil told the inspiring story of Ramanujan to the Japanese mathematicinas, who were not familiar with him [25]. It was at this Conference that Yutaka Taniyama first stated the famous Taniyama-Shimura-Weil Conjecture, about elliptic curves and modular forms. [26] After almost 3 decades, Ken Ribet [24] made a breakthrough by proving the Epsilon Conjecture, first proposed by Serre, thereby establishing that the Taniyama-Shimura-Weil Conjecture implies the Fermat's Last Theorem, and the history. The work at the 1955 Conference, has an important role to play in Wiles's famous proof. Thus, the Tokyo 1955 Conference has an important place in the history of Number Theory.

Today, the School of Mathematics, Tata Institute of Fundamental Research (TIFR), Mumbai is a leading centre

Figure 3. KGR (second from right) with (from left) M. S. Raghunathan, S. Chandrasekhar (Nobel Laureate in Physics, 1983), B. V. Sreekantan, M. S. Narasimhan and K. Ramachandra, during Chandrasekhar's 1987 visit to the Institute.
Courtesy - **TIFR Archive**.

of pure mathematics research with brilliant faculty and international reputation, and this would never have been possible without the effort of the two. A visionary, Ramanathan realized the need for establishing a centre for Applied Mathematics in India. The idea of establishment of a joint TIFR-IISc programme (which is the TIFR Centre for Applicable Mathematics (CAM)) to be operated from the campus of the Indian Institute of Science, Bangalore in 1975 was his brainchild. In fact, he preferred the phrase "applicable" to "applied". Today, the TIFR-CAM has become an important Institution, where faculties are actively working on many aspects of differential equations and numerical methods. In February 2005, an Indo-French Workshop on Partial Differential Equations and their Applications was organized at the Department of Mathematics, IISc Bangalore [22]. Several distinguished speakers form both the Indian as well as French math community participated in it. This event was dedicated to the memory of KGR and **Prof. Jacques-Louis Lions** (father of Pierre Louis Lions, Fields Medal, 1994), who were pioneers in initiating the Indo-French cooperation in Applied Mathematics in the 1970s. Thus, Ramanathan holds the distinction of leading the foundation of both pure as well as applied mathematics schools in India [14]. Such excellent initiative can indeed be compared with that of Prasanta Chandra Mahalonbis and C.R Rao in setting up the renowned school of Statistics and Probability at the Indian Statistical Institute, Kolkata and later in its branches in Bengaluru and Delhi. Almost 7 decades have passed after the establishment of the TIFR, and today we have so many other good places to teach and do mathematical research in India. During his time as at the Tata Institute in Mumbai, Ramanathan closely and enthusiastically interacted with numerous research scholars during their long walks along the Arabian Sea, and with his vast knowledge, exposed them to many deep areas of pure mathematics, which were still not that much popular in India. He also had a nice sense of humour and had the habit of sharing many interesting anecdotes about other mathematicians among his students. An excellent teacher and expositor of the subject, Ramanathan's lessons and guidance had a great impact on their future mathematical career. Some of the students went on to become acclaimed mathematicians, like MS Narasimhan, C S Seshadri and R Sridharan [10] (all three of them were Shanti Swarup Bhatnagar Awardees). An interesting anecdote is worth making in this regard. After completing their Doctoral degree in India (where they were mostly self taught), both Narasimhan and Seshadri had went for Postdoctoral stints in Paris, with the former being mentored by Laurent Schwartz (Fields Medal, 1950) and the latter by Claude Chevalley [15]. Even before they went to Paris, it was KGR who drew their attention to a very important paper of Weil on vector bundles on compact Riemann surface (which did not directly belong to Ramanathan's own area of research), titled "Generalisation des fonctions abeliennes". KGR was himself made aware of this work of Weil by Seigel. This was played a very crucial role in their research, as they got influenced and began working on the topic. In 1964, they finally established their celebrated result, the **Narasimhan Seshadri Theorem**, which concerns the stability of holomorphic vector bundles over a Riemann Surface.

3. Mathematical Works

Now we would discuss some mathematical works of Ramanathan. Broadly speaking, his interests were primarily in Number Theory and related fields. He was extremely interested in exploring and extending the published and unpublished works of Ramanujan and had a few publications on the same. He was also actively involved in studying his Notebooks, and wrote some papers [8], after being motivated by the beautiful mathematics of Ramanujan, which were mainly based on the topics of congruence properties of some arithmetical functions, Ramanujan's Trigonometric Sums and Ramanujan type identities. One of his remarkable papers was "Identities and Congruence of the Ramanujan Type" [16], published in 1950. We know that Ramanujan was famous for his work on $P(n)$, which denotes the number of "unrestricted **partitions**" of a natural number n. For example, $P(5) = 7$, i.e., there are 7 partitions of 5, namely - $(4 + 1)$, $(3 + 2)$, $(3+1+1+1)$, $(2+2+1)$, $(2+1+1+1)$ and $(1+1+1+1+1)$. One may notice that we are trying to write n as sum of positive integers $\leq \ n$, where the summands are in **non-increasing** order. Partition is an important object of study in Number Theory. Below, we state one of Ramanathan's result from that paper:

Theorem 3.1. *Let $v > 0$,*

$$\sum_{n=0}^{\infty} P_v(n)x^n = \frac{1}{[(1-x)(1-x^2)\cdots]^{-v}}$$

The coefficients $P_v(n)$ satisfy the following properties:

(1) If $24\, m \equiv v (mod\ 5^a)$, then $P_v(m) \equiv 0$:

- *(a)* $\left(\text{mod } 5\right)$ *if* $v \equiv 12, 17, 22, 27 \pmod{30}$
- *(b)* $\left(\text{mod } 5^{[\frac{a-1}{2}]}\right)$ *if* $v \equiv 15, 20, 25 \pmod{30}$
- *(c)* $\left(\text{mod } 5^{[\frac{a}{2}]}\right)$ *if* $v \equiv 3, 4, 8, 9 \pmod{30}$
- *(d)* $\left(\text{mod } 5^{[\frac{a+1}{2}]}\right)$ *if* $v \equiv 16, 21, 26 \pmod{30}$
- *(e)* $\left(\text{mod } 5^{a-1}\right)$ *if* $v \equiv 0, 5, 10 \pmod{30}$
- *(f)* $\left(\text{mod } 5^{[\frac{a+2}{2}]}\right)$ *if* $v \equiv 2, 7 \pmod{30}$
- *(g)* $\left(\text{mod } 5^{a}\right)$ *if* $v \equiv 1, 6, 11 \pmod{30}$

(2) If $24\, m \equiv v (mod\ 7^a)$, then $P_v(m) \equiv 0$:

- *(a)* $(\text{mod } 7)$ *if* $v \equiv 8, 15, 22 \pmod{28}$
- *(b)* $(\text{mod } 7^{[\frac{a-1}{2}]})$ *if* $v \equiv 14, 21 \pmod{28}$
- *(c)* $(\text{mod } 7^{[\frac{a}{2}]})$ *if* $v \equiv 2, 3, 5, 6 \pmod{28}$
- *(d)* $(\text{mod } 7^{[\frac{a+1}{2}]})$ *if* $v \equiv 11, 18, 25 \pmod{28}$
- *(e)* $(\text{mod } 7^{a-1})$ *if* $v \equiv 0, 7 \pmod{28}$
- *(f)* $(\text{mod } 7^{[\frac{a+2}{2}]})$ *if* $v \equiv 1 \pmod{28}$
- *(g)* $(\text{mod } 7^{a})$ *if* $v \equiv 4 \pmod{28}$

Here, [.] denotes the usual "greatest integer function." The proof of Ramanathan uses some techniques developed earlier by mathematicians H. Rademacher (specially construction of some special functions) and G. N. Watson, and uses the Dedekind Modular form, defined as follows:

$$\tau(\eta) = e^{\frac{i\pi\tau}{12}} \prod_{n=1}^{\infty} (1 - e^{2n\pi i\tau})$$

and the "hauptmodul" $\phi(\tau)$ of a modular function The proof also rests on **8 lemmas**, where some new entire modular functions are defined and manipulated accordingly. In particular, the following function is constructed:

$$F_{n,v} = \phi(\tau)^{kt} \sum_{i>0} a_i(n) p^{i-1} \phi(\tau)^i$$

(the $a_i(n)$ being integers depends on ,n, p and v) used with other functions for proving the congruence properties for $P_v(n)$.

Ramanujan had conjectured that:

$$P(n) \equiv 0 \ (\text{mod } 5^a 7^b 11^c)$$

It is not known whether this is true for all c. Though he did not prove these, he indicated that these congruences may be deduced from identities of this kind:

$$P(4) + P(9)x + \cdots = 5 \cdot \frac{[(1-x^5)(1-x^{10})\cdots]^5}{[(1-x)(1-x^2)\cdots]^6}$$

The above theorems established by Ramanathan for are analogous to these identities of Ramanujan, and can be applied to study them in details, and perhaps derive some similar formulas. Ramanujan considered the following sum in one of his papers published in 1918 [2]:

$$c_q(n) = \sum_{\substack{1 \leq a \leq q \\ (a,q)=1}} e^{\frac{2\pi i a n}{q}}$$

$c_q(n)$, usually called the Ramanujan Sum, is a function of two positive integer variables q and n. Ramanathan studied this function in great details [3,4] and found many new properties. Now, if a natural number n has the following property:

$$n \equiv e_1 + e_2 + \cdots \pmod{m}$$

then n is said to be relatively partitioned (mod m). von Sterneck used a certain two variable function $f(m,n)$ to study and write many formulas related to this theory in closed form. Ramanathan showed that $f(m,n)$ is same as that of the Ramanujan Sum $c_q(n)$. He also found many other applications of Ramanujan's Trigonometric sum in the theory developed by von Sterneck. He had a famous paper [5] on the applications of Kronecker's limit formula. Ramanathan also did important work in other areas of Number Theory, namely Diophantine Inequalities and automorphic functions related to Siegel's formula. He also generalised certain results of mathematicians Hecke and Maass. KGR also extensively worked on the properties of unit groups of quadratic and Hermitian forms over algebraic number fields [6,7] with one of his papers coming out in the *Annals of Mathematics*. In modern number theory, arithmetic groups are an important class of objects, which are of interests to Topologists as well. It was KGR who popularized the study of Arithmetic Groups in India, and later it became one of the main research topics at the Tata Institute. His work on the analytic and arithmetic theory of quadratic forms over involutorial division algebra is internationally recognized. KGR, along with Raghavan also worked and contributed significantly towards the famous Oppenheim Conjecture [18], which concerns quadratic forms in many variables. The conjecture was finally settled in the affirmative by Margulis in 1987. Coauthoring a paper with the Indo-Canadian mathematician Mathukumalli V. Subbarao [13] gave him an Erdős number of **2**.

Interested readers may check out an obituary written by Prof. S. Raghavan [11] where some other aspects of his works (mainly involving discrete groups) are covered. It also contains the complete list of **48** research papers and **2** books of Ramanathan.

4. Awards, Honours and Recognition

Throughout his career, Ramanathan received numerous honours and recognition. He was awarded the coveted Shanti Swarup Bhatnagar Prize in Mathematical Science (one of the highest recognitions for an Indian researcher) in the year 1965, the Padma Bhushan, the third highest civilian award awarded by the Government of India in 1983 and the Homi Bhabha Medal of the Indian National Science Academy (INSA) in 1984. Ramanathan was the Fellow of the Indian Academy of Sciences, Bangalore, Indian National Science Academy, New Delhi, the Jawaharlal Nehru Fellowship and the Foundational Fellow of the Maharashtra Academy of Sciences. He was on the Editorial Board of the prestigious journal Acta Arithmetica for almost three decades, and was the Editor of the Journal of the Indian Mathematical Society for more than a decade. He also served as the President of the prestigious Society. During his 34 year old association with the TIFR, Ramanathan also visited other centres of mathematics, like the Institute for Advanced Study at Princeton, the University of Missouri at St. Louis and the University of Alberta at Edmonton, to name a few. He had authored two books. He was also an influential mentor, guiding several distinguished students, many of whom continued to work in the Math Department of TIFR. Below listed are some of his doctoral students. The list has been obtained from an excellent review article (based on a lecture given at IMSc, Chennai) [17] by the eminent Indian mathematician, Prof. Dipendra Prasad.

- C. P Ramanujam – He was a talented mathematician, working in the areas of number theory and algebraic geometry. Like his namesake Ramanujan, he too died early after suffering from illness.
- K. Ramachandra – He was a leading figure of Analytic, Algebraic and Transcendental Number Theory in India, well known for his work on the Riemann Zeta function and the six exponential theorem. His doctoral students include influential Number Theorists R. Balasubramanian and T. N. Shorey.
- S. Raghavan – Like his mentor KGR, he was also a winner of the prestigious Shanti Swarup Bhatnagar Award and was interested in the mathematics of Ramanujan. During his time at TIFR, he was the Dean of the School of Mathematics. From 1978–1982, he served as the Editor of the prestigious journal Proceedings(Mathematical Sciences), published by the Indian Academy of Sciences. It is also worth mentioning that one of his papers was the first from TIFR to appear in the Annals of Mathematics [9].
- Neela S. Rege – Born on 7th November 1941 in Baroda, Gujarat, she completed her Bachelors and

Masters from the from the University of Bombay in 1963, and subsequently earned her PhD from TIFR in 1968. She received the G. Rangildas Mathematics prize in 1961 and the P. Thackersey award in 1963 from the University of Bombay. Dr. Rege is member of many learned Societies, like the American Mathematical Society, London Mathematical Society and Ramanujan Mathematical Society. During 1975–77, she was a Visiting Professor at U. dakar, Senegal. Since 1989, she is a Senior Lecturer at P.N.G University of Technology, Lae, Morobe, Papua New Guinea. [21]

- Sunder Lal – Born on 11th October,1934 he received his MA (Mathematics) from Panjab University in 1958 and PhD degree from TIFR under the supervision of KGR in 1965, working in the area of one variable Modular forms. He joined the Department of Mathematics at Punjab University as a Reader on 13th July, 1966, and served there for about three decades, retiring in October 1994. [20]
- V. C. Nanda – Like Dr. Lal, he was also a Professor of Mathematics at Punjab University.
- S. S. Rangachari – He was an eminent Number Theorist, working at the Tata Institute of Fundamental Research in Mumbai. He was interested in the works of Ramanujan. Post retirement, he had settled down in the United States of America.

5. Later Life

Apart from academics, Ramanathan was deeply interested in music and was a singer himself. He was also fond of English, Tamil and Telugu literature. When he was at Princeton, for about a couple of years, his neighbour was none other than Albert Einstein. He used to sing Carnatic songs to Einstein for his entertainment, especially songs of Tyagaraja. As a person, Ramanathan was a simple and kind person, who could be easily approached by students. He was married to Jayalakshmi Ramanathan, and the couple had 2 sons, Ananth and Mohan and 4 grandchildren Aparna, Kavitha, Anita and Nikil. In December 1985, Ramanathan retired from the Tata Institute in Mumbai (erstwhile Bombay). During his last few years, he suffered from Parkinson's disease, and also underwent a cerebral surgery. After suffering for a long time, Ramanathan passed away on May 10, 1992 at the age of 72, marking an end to a very significant chapter in the history of Indian mathematics. The Proceedings (Mathematical Sciences) published a special issue in his honour, titled "K. G. Ramanathan Memorial Issue". Renowned mathematicians from all over the world (including some of KGR's own students and grandstudents), including Roger Heath Brown FRS, Raman Parimala, Don Zagier, R. P. Bambah, R. Balasubramanian, Bruce C. Berndt, Richard Askey, George C. Andrews contributed papers, and the issue received a review from another leading Indian mathematician, M.S Raghunathan FRS [12]. Thus, K. G. Ramanathan was a luminous figure in Indian mathematics post independence, and played a decisive role in bringing us back to the international mathematical map. A front ranked mathematician, a legendary institution builder and a great human being, KGR's legacy continues even till this date, and would definitely inspire the future generations of Indian students to take up a life long career in mathematics.

6. Acknowledgement

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Notes on the Domain of Exponent Pairs

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Abstract. The theory of exponent pairs as initiated by Phillipps in 1933 proposes pairs of exponents (κ, λ) so that one has $\sum_{n\sim N} e^{2i\pi\varphi(n)} \ll_\varepsilon F^{\kappa+\varepsilon}N^{\lambda+\varepsilon}$, for any positive ε, where φ is a 'monomial-like' smooth function whose first derivative is of size about F. We propose to explore the domain of available pairs (κ, λ) through a very geometrical approach. We prove in particular that this domain is the convex hull of a connected curve in the classical case. We also show that a possible choice for λ, for any $\kappa \in [0, 1/2]$, is given by $\lambda = 1 - \frac{\kappa}{\log 2} \log \frac{2\kappa+1}{2\kappa}$. We finally recall rapidly how this theory has been adapted to the higher dimensional setting. In passing, we take the opportunity of this slow-paced paper to describe some usage of the SageMath software.

Keywords. Exponential sums, Exponent pairs.

AMS Mathematics Subject Classification: 11L03.

1. Introduction and Results

Exponent pairs in the large. E. Phillipps developped in [8] a theory of *exponent pairs* by furthering and simplifying the notion of *exponent system* introduced by J. G. van der Corput in [12]. The reader will find a modern account of this theory in the reference book [6] by S. W. Graham and G. Kolesnik. Roughly speaking a couple $(\kappa, \lambda) \in \left[0, \frac{1}{2}\right] \times \left[\frac{1}{2}, 1\right]$ is said to be *an exponent pair* when, given a regular function φ that is 'monomial-like' and whose first derivative on the interval $[N, 2N]$ is of size F, the upper bound

$$S = \sum_{N<n\le 2N} e^{2i\pi\varphi(n)} \ll_\varepsilon F^{\kappa+\varepsilon}N^{\lambda+\varepsilon}.$$

holds for any $\varepsilon > 0$. The following exponent pairs are known:

$$(0,1), \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{6}, \frac{2}{3}\right), \left(\frac{1}{14}, \frac{11}{14}\right), \left(\frac{9}{56}, \frac{37}{56}\right), \left(\frac{89}{560}, \frac{369}{560}\right). \tag{1}$$

An example. In [2], Dekking and Mendès-France propose a geometrical approach to exponential sums. The reader will find there several examples and some compelling drawings. A most classical example is $\varphi(n) = t(\log n)/(2\pi)$ where t is some large parameter and for instance $N = t^{1/3}$. The sum S is then often called a *zeta-sum*. A first trivial bound for S is $N+1$. Since $\varphi'(n) = t/(2\pi n)$, we see that $t^{2/3} < 4\pi\varphi'(n) \le 2t^{2/3}$, so that we may select $F = N^2$. We thus get the bound $S \ll_\varepsilon N^{2\kappa+\lambda+\varepsilon}$, and with the pairs given in (1) above, this gives the exponents

$$1, \frac{3}{2}, 1, \frac{13}{14} = 0.928\ldots, \frac{55}{56} = 0.982\ldots, \frac{547}{560} = 0.976\ldots \tag{2}$$

of which $13/14$ is the best one for our problem.

We encourage the readers to find the best bound obtainable in shorter ranges, and for instance when $N = t^{1/5}$.

The domain of exponent pairs. It follows from the theory that, given an exponent pair (κ, λ), we may build another one by the two formulas

$$\left(\frac{\kappa}{2\kappa+2}, \frac{\kappa+\lambda+1}{2\kappa+2}\right), \quad \left(\lambda - \frac{1}{2}, \kappa + \frac{1}{2}\right). \tag{3}$$

The first five pairs above are obtained by using these two processi while the last two have been obtained respectively by M. N. Huxley and N. Watt in [7] and by N. Watt in [13]. Furthermore, any convex combination of exponent pairs is again an exponent pair. When using the pair $(0, 1)$ and the processi described in (3), we call the convex hull of the domain obtained the *van der Corput Domain* denoted by $\mathscr{D}$.

Theorem 1.1. *The domain $\mathscr{D}$ is the convex hull of the curve $\mathcal{C}$ defined in* (9) *and which is the graph of a continuous non-increasing function.*

When we add the point $\left(\frac{89}{560}, \frac{369}{560}\right)$, the domain will be called the *Watt Domain* in the sequel and denoted by $\mathscr{D}^*$.

In [9], R. A. Rankin started to describe the set of accessible exponent pairs, a study furthered by S. W. Graham in [5]. The viewpoint taken in both papers is to compute optimal values in a specific problem. The aim of the present note is to continue this work from a more geometric viewpoint. Nonetheless it is fair to say that a large part of the material we present here can be found in the previous two papers in some form or some other.

Since we also strive to describe the situation with pictures, it may be better to provide the readers with the means to play themselves with these pictures. We shall be using SageMath, see [11]; the script we use is available on the web at:

`https://ramare-olivier.github.io/Maths/ExpPairsNote-01.sage`

Copy this code is a file named, say, `ExpPairs.sage`, without forgetting the `sage` suffix, start SageMath and load this via the command `load("ExpPairs.sage")`. We give in the text some pointers on to how to code in SageMath, as well as commands that we write in the form

```
ExpPairs.sage/plotC(12, 6)[1]
```

to mean that the reader should type the command `plotC(12, 6)[1]` in SageMath, once the main file `ExpPairs.sage` has been duly loaded.

A simple continuous bound. A consequence of our study is the next flexible estimate.

Theorem 1.2. *Let S be an exponential sum of monomial type and parameters N and F. Then, for every $\kappa \in \left[0, \frac{1}{2}\right]$ and every $\varepsilon > 0$, we have $S \ll_\varepsilon F^{\kappa+\varepsilon} N^{\vartheta_0(\kappa)+\varepsilon}$ where*

$$\vartheta_0(\kappa) = 1 - \frac{\kappa}{\log 2} \log \frac{2\kappa + 1}{2\kappa}.$$

We have $\vartheta_0(0) = 1$, $\vartheta_0(1/2) = 1/2$ and $\vartheta_0(1/6) = 2/3$.

This is proved in Lemma 2.8 below. The upper bound $S \ll_\varepsilon F^{\varphi(\lambda+\frac{1}{2})-\frac{1}{2}+\varepsilon} N^{\lambda+\varepsilon}$ also holds true for every $\lambda \in \left[\frac{1}{2}, 1\right]$ and every $\varepsilon > 0$, and this one is better than the above one when $\lambda \leq 2/3$. Theorem 5.1 belows offers a generalization of this result to higher dimensional exponential sums. Contrarily to S. W. Graham's approach that leads to optimal values at some specific points, Theorem 1.2 allows real-valued optimization at a small numerical loss in the exponent. The approximation is however very tight as shown by Figure 1.

On our example, we get $S \ll_\varepsilon N^{2\kappa+\vartheta_0(\kappa)+\varepsilon}$, which is minimal when $\kappa = 0.0566\,865\ldots$ with value $N^{0.926\ldots}$, whence

$$\sum_{t^{1/3} < n \leq 2t^{1/3}} n^{it} \ll t^{0.926/3} \tag{4}$$

improving on (2), though still far from the expected $t^{\frac{1}{6}+\varepsilon}$.

Preparing for the proofs: a change of variables. We prefer to change of variables and to use

$$(u, v) = (2\kappa, 2\lambda - 1). \tag{5}$$

The pairs above become $(0, 1)$, $(1, 0)$, $(\frac{1}{3}, \frac{1}{3})$, $(\frac{1}{7}, \frac{4}{7})$, $(\frac{9}{28}, \frac{9}{28})$, $(\frac{89}{280}, \frac{89}{280})$, while the transformations (3) read

$$f(u, v) = \left(\frac{u}{u+2}, \frac{v+1}{u+2}\right), \quad c(u, v) = (v, u). \tag{6}$$

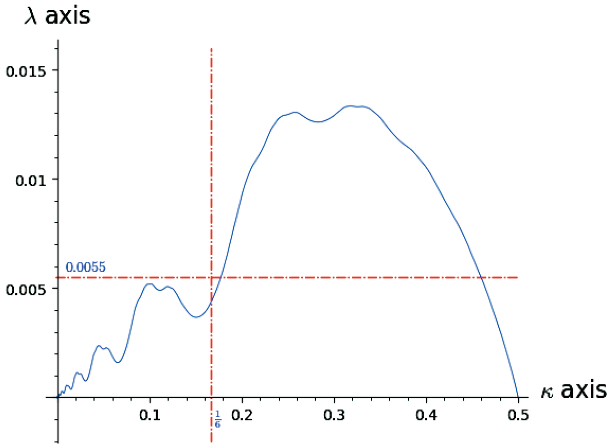

Figure 1. Difference between ϑ_0 and the optimal exponent pair from the van der Corput domain, drawing obtained via `ExpPairs.sage/compareModelConvHull(12)`.

As c is an involution, it is better to consider the transform $g = c \circ f$ and to consider iterations of f and g. It is noteworthy that f and g preserve segments. One can consider these transforms as restrictions of linear transforms on the projective plane $\mathbb{P}_2$:

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 2 \end{pmatrix} \tag{7}$$

so that $f(u, v)$ can be read on the first two coordinates of $C \begin{vmatrix} u \\ v \\ 1 \end{vmatrix}$ (when we divide by the third one); a similar link holds between g and D.

2. A First Player: The Curve $\mathcal{C}$

There are two ways to describe the van der Corput Domain. Both rest on a curve $\mathcal{C}$ that we now build: either by taking a limit from above, or by taking the closure of the set of points obtained by iterating f and g when starting from $\{(0, 1), (1, 0)\}$.

Getting to $\mathcal{C}$ from outside. The construction we now describe will for instance make clear that we reach a connected curve.

In this section, we consider the transformation of the unit square $[0, 1]^2$ under the two transforms f and g. Let us first notice that these transforms are contracting.

Lemma 2.1. *When $P, Q \in [0, 1]^2$, we have $\|f(P) - f(Q)\|_2 \leq \rho \|P - Q\|_2$ where $\rho = \sqrt{\frac{3+\sqrt{5}}{8}} \leq 13/16$. The same holds true for g.*

Proof. The Jacobian reads, with $U = u + 2$ and $V = v + 1$,

$$J = \begin{pmatrix} \frac{2}{U^2} & -\frac{V}{U^2} \\ 0 & \frac{1}{U}, \end{pmatrix} \quad \text{so that} \quad U^4 JJ^* = \begin{pmatrix} 4+V^2 & -UV \\ -UV & U^2 \end{pmatrix}.$$

The largest eigenvalue of JJ^* is

$$\lambda = \frac{4 + U^2 + V^2 + \sqrt{U^4 + (2V^2 - 8)U^2 + (4 + V^2)^2}}{2U^4}.$$

It is largest when $V = 2$, so we are left with finding the maximum over $W \in [4, 9]$ of the quantity

$$\frac{8 + W + \sqrt{64 + W^2}}{2W^2}.$$

As this function of W is non-increasing, the worst case is $W = 4$. The lemma follows readily. □

Let $\mathcal{K}([0, 1]^2)$ be the compact space of the compact subsets of $[0, 1]^2$, equipped with the usual Hausdorff distance (see for instance Exercise 3 of Section 16, Chapter 3 of the reference book [3] by J. Dieudonné), i.e.

$$d(K_1, K_2) = \max\Big(\max_{k_1 \in K_1} d(k_1, K_2), \max_{k_2 \in K_2} d(k_2, K_1)\Big). \quad (8)$$

We, rather obviously, still call f the function induced by f on $\mathcal{K}$. And $f \cup g$ is the function that, to any set $\mathcal{A}$, associates $f(\mathcal{A}) \cup g(\mathcal{A})$. This is a continuous function. We set

$$\mathcal{C} = \bigcap_{n \geq 0} (f \cup g)^{\circ n}([0, 1]^2) \quad (9)$$

where notation $(f \cup g)^{\circ n}$ means that we compose $(f \cup g)$ iteratively n times with itself. This set $\mathcal{C}$ corresponds to the set $\mathscr{P}$ of S. W. Graham in [5]. Plotting $\mathcal{C}$ is not difficult.

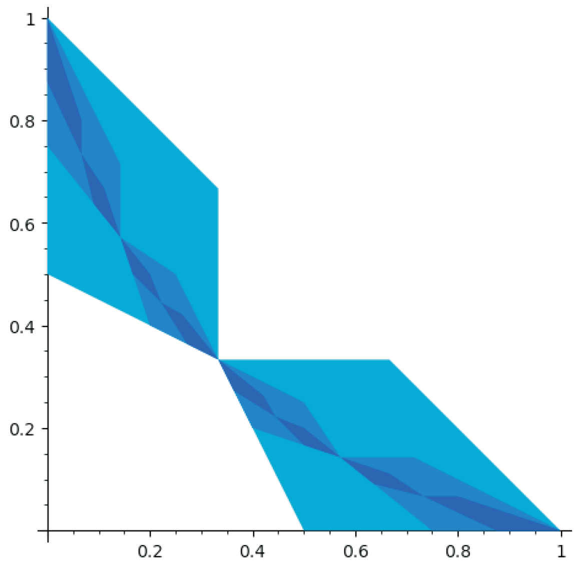

Figure 2. The first three iterations, drawing obtained via `ExpPairs.sage/transformSquare([[0,0],[0,1],[1,1],[1,0]], 3,2,False)`.

Here are some details on the SageMath code that produces it.

```
######### Common handlers ##################
def fC(p):
    return([p[0]/(p[0]+2), (p[1]+1)/(p[0]+2)])

def gD(p):
    return([(p[1]+1)/(p[0]+2), p[0]/(p[0]+2)])

######### Building of the C-curve from up ######
def actonD(InitDom, nbsteps, myz, doplot = True,
           shade = 0):
    # When doplot = False, be sure nbsteps > 0
    acolor = Color(0/255, (70 + 40*(nbsteps + shade))/255,
    200/255)
    DomfC = list(map(fC, InitDomain))
    DomgD = list(map(gD, InitDomain))
    ToPlot = Graphics() # empty graphical object
    if doplot:
        ToPlot = polygon(InitDom, color = acolor,
                         zorder = myz)
        if nbsteps > 0:
            ToPlot += actonD(DomfC, nbsteps-1, myz+1, True,
                             shade)
            ToPlot += actonD(DomgD, nbsteps-1, myz+1, True,
                             shade)
    else:
        if nbsteps > 0:
            ToPlot = actonD(DomfC, nbsteps-1, myz+1, True,
                            shade)
            ToPlot += actonD(DomgD, nbsteps-1, myz+1, True,
                             shade)
    return(ToPlot)

MyInitialDomain = [[0,0], [0,1], [1,1], [1,0]]
actonD(MyInitialDomain, 6, 2, False, 110).show(figsize = 15,
       gridlines = "automatic", xmin = 0, xmax = 1,
       ymin = 0, ymax = 1, aspect_ratio = 1)
```

Copy this code is a file `CCurve.sage`, without forgetting the `sage` suffix, start SageMath and run the file by using the command `load("CCurve.sage")`. It is then possible to increase `nbsteps`, say to 10, and to zoom on a particular region by changing the quadruple `(xmin, xmax, ymin, ymax)`. Let us take the opportunity of this note to explain part of the help system of Sagemath. If we set `P = actonD(MyInitialDomain, 1, 1, False, 0)`, then we may use `P.<tab>` to get access to all the methods associated with the object P (a plot). And to see all the tons of options associated with the `show` method, enter `P.show?`. The 2D-plotting reference guide is available there:

https://doc.sagemath.org/pdf/en/reference/plotting/plotting.pdf

Let us comment on this picture. Introducing an adhoc definition will simplyfy our task.

Definition 2.2. *A* tile *is the convex hull of four points.*

Since the functions f and g transform segments into segments, tiles are transformed into tiles by any composition-product of these two.

We start from the tile $[0,1]^2$, which we transform by f, getting the new tile $\mathrm{Conv}\left((0,1),\left(\frac{1}{3},\frac{2}{3}\right),\left(\frac{1}{3},\frac{1}{3}\right),\left(0,\frac{1}{2}\right)\right)$ and then similarly by g, getting the tile $\mathrm{Conv}\left(\left(\frac{1}{3},\frac{1}{3}\right),\left(\frac{2}{3},\frac{1}{3}\right),(1,0),\left(\frac{1}{2},0\right)\right)$. These resulting tiles join in $\left(\frac{1}{3},\frac{1}{3}\right)$. On applying repeatedly the transforms f and g, we get a connected necklace of tiles. Here is a lemma that helps structure the situation.

Lemma 2.3. *Let K be a connected compact subset of $[0,1]^2$ that contains the points $(0,1)$ and $(1,0)$. The set $f(K) \cup g(K)$ is again a connected compact subset of $[0,1]^2$ that contains the points $(0,1)$ and $(1,0)$.*

Proof. Indeed $f(K)$ and $g(K)$ are both connected and compact. Both sets contain the point $f((1,0)) = \left(\frac{1}{3},\frac{1}{3}\right) = g((0,1))$, so that $f(K) \cup g(K)$ is connected. It contains the points $f((0,1)) = (0,1)$ and $g((0,1)) = (1,0)$. □

At each step, we get a succession of tiles $h(0,1) - h(1,1) - h(1,0) - h(0,0)$; the distance between $h(1,1)$ and $h(0,0)$ is at most $\rho^n \sqrt{2}$, when h is a product of n terms from $\{f,g\}$. This shows an exponential rate, and the actual rate is faster (meaning the practical 'ρ' is smaller). One shows readily that we end up with a curve. To plot it, we may only consider the transforms of the lower part of the initial square. Let us state formally a theoretical consequence of this discussion.

Lemma 2.4. *The curve $\mathcal{C}$ is the graph of a continuous non-increasing function.*

We shall now see that this seemingly regular curve $\mathcal{C}$ contains a dense subset of *rational* points (i.e. points whose coordinates are rational numbers).

Getting to $\mathcal{C}$ from inside, I. We may get points that are *on* the final curve by two processi. Here is a first one.

The construction above shows also that each point of $\mathcal{C}$ may be reached in a unique manner either from any point with a *infinite* sequence of f and g, giving an adapted 'binary' writing for these points. For instance the point $(1,0)$ is $fff\ldots$, while $(0,1)$ is $ggg\ldots$: indeed, the first f reduces the unit square $[0,1]^2$ to a smaller parallelepiped that lies inside $[0,1/3]\times[1/3,1]$. On applying f again, we get an even smaller parallelepiped that still contains the point $(0,1)$. The intersection

$$\bigcap_{n\ge 1} f^{\circ n}([0,1]^2)$$

then reduces to the point $(0,1)$. Please notice that when we associate the sequence $fff\ldots$ to this points, the order is reverse to the one we use for the composition of functions.

The point $\left(\frac{1}{3},\frac{1}{3}\right)$ is $gffff\cdots = fgggg\ldots$. The same construction shows that any finite combination of f and g applied to $(0,1)$ or $(1,0)$ belongs to $\mathcal{C}$. We get rational points by considering a finite sequence, say $ffgfg$, and completing it on the right either by $fff\ldots$ if we want to refer to the upper left point of the parallelepiped $gfgff([0,1]^2)$, or by $ggg\ldots$ if we want to refer to the lower right point of the same parallelepiped. We get in this manner rational points that are on the curve $\mathcal{C}$. The reader will readily see that we get a dense family of such points. Indeed, specifying a prefix, like $ffgfg$, localizes the point inside $gfgff([0,1]^2)$ and any continuation, say $ffgfgffggffgg$, leads to points that are inside this set.

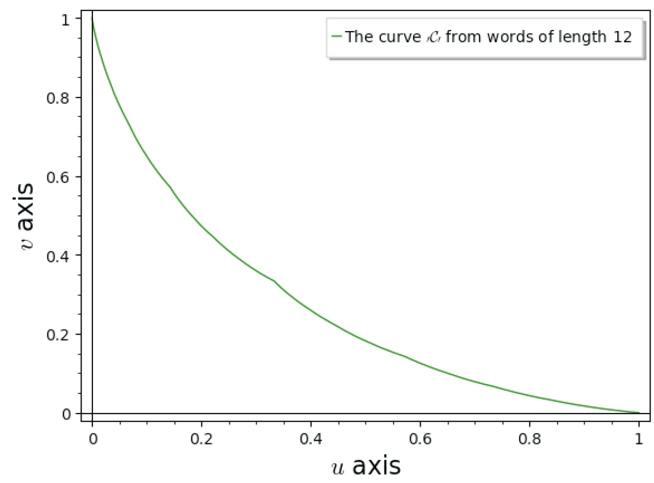

Figure 3. Approximation of the curve $\mathcal{C}$ with words of length 12, drawing obtained through `ExpPairs.sage/plotC(12, 6)[1]`.

Getting to $\mathcal{C}$ from inside, II. We now describe a second process to get points that are *on* the final curve.

A point of the curve may be attained by some sequence, say $ffgfg\ldots$. Rather than considering the transforms f and g, we could equivalently make the corresponding product of matrices C and D. These and the resulting product is a non-singular matrix with (integer) non-negative coefficients. As such it has a single dominant eigenvalue, the so-called Perron-Frobenius (see Chapter XIII of the book [4] by

Gantmacher) eigenvalue and a corresponding eigenvector. So, if we iterate the transform $g \circ f \circ g \circ f \circ f$, or equivalently $DCDCC$, the image in $\mathbb{P}_2$ of the cone corresponding to the square $[0,1]^2$ accumulates around the line containing this eigenvector. This line is indeed a point of $\mathcal{C}$; it corresponds to the code $ffgfgffgfgffgfg\ldots$ where we repeat the pattern $ffgfg$. The points we now obtain are cubic, since so is the eigenvalue as a root of a cubic polynomial, namely the characteristic polynomial (of $DCDCC$ in our example). Here again, we can localize these points by choosing a proper prefix.

Additional properties. Here are three additional properties of $\mathcal{C}$.

Lemma 2.5. *If $f(P)$ belongs to $\mathcal{C}$, then P belongs to $\mathcal{C}$.*

Proof. We may assume that $f(P) \neq \left(\frac{1}{3}, \frac{1}{3}\right)$. Indeed, if $f(P)$ is in $\mathcal{C}$, then $f(P)$ is a limit point of a sequence $f^{a_1} g^{a_1} f^{a_2} g^{a_2} \ldots$. As $f(P) \neq \left(\frac{1}{3}, \frac{1}{3}\right)$ and the image of $[0,1]^2$ by f and g only intersect on this point, we deduce that $a_1 > 0$. By injectivity, we see that $P = f^{a_1 - 1} g^{a_1} f^{a_2} g^{a_2} \ldots$, completing the proof. □

Lemma 2.6. *The three areas that are (1) the points that are strictly above $\mathcal{C}$, (2) the points that are on $\mathcal{C}$ and (3) the points that are strictly below $\mathcal{C}$ are stable under the action of f and g.*

Proof. Indeed f and g are injective maps. Let $P = (u, v)$ be above $\mathcal{C}$. This means that the segment $[P, P_0]$ where $P_0 = (u, 1)$ does not cross $\mathcal{C}$. The segment $[(0,1),(1,1)]$ on which P_0 lies remains above our curve by construction after applying f or g. The segment $[f(P), f(P_0)]$ may not cross $\mathcal{C}$, as a crossing point would be a $f(Q)$, and by Lemma 2.5, Q would belong to $\mathcal{C}$ and to $[P, P_0]$, a contradiction. □

Lemma 2.7. *If a point P is below (resp. up of) the curve $\mathcal{C}$, then $f(P)$ and $g(P)$ are also there.*

A simple continuous bound. We now present a readily exploited upper bound for $\mathcal{C}$.

Lemma 2.8. *The graph of the function $\theta_0 : x \mapsto 1 - x \log(1 + 1/x)/\log 2$ is stable under f and remains above $\mathcal{C}$. It crosses $\mathcal{C}$ in three points:* $(0, 1)$, $\left(\frac{1}{3}, \frac{1}{3}\right)$ *and* $(1, 0)$.

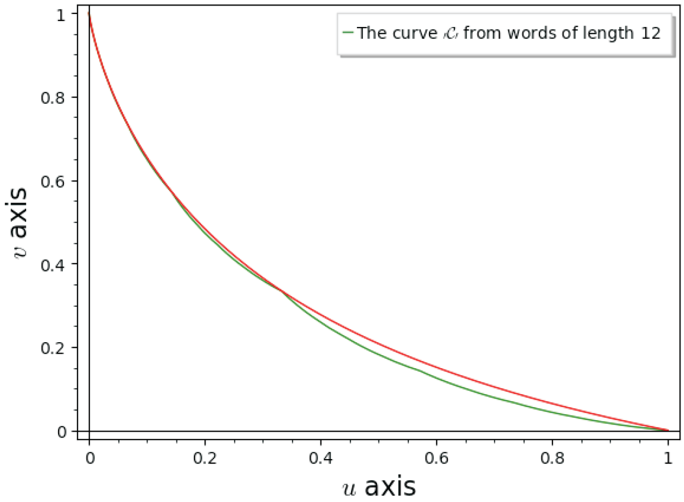

Figure 4. The curve $\mathcal{C}$ in green (below) and the graph of θ_0 in red (above), drawing obtained through `ExpPairs.sage/plotC(12, 6)[1]+plotUpper(6)`.

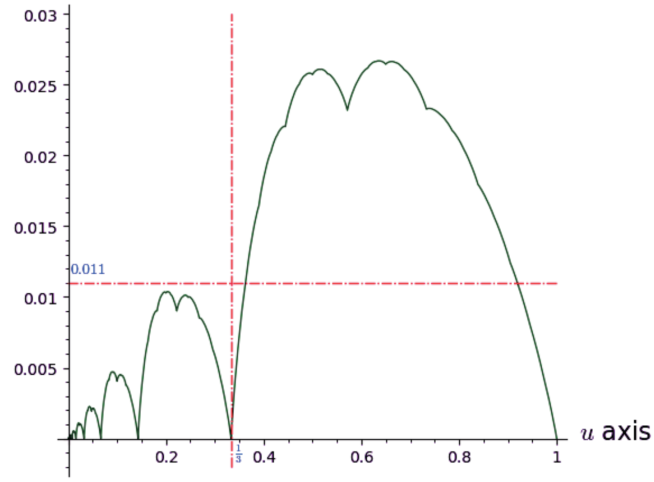

Figure 5. Difference between θ_0 and $\mathcal{C}$, drawing obtained by using `ExpPairs.sage/compareModelCurve(12)`.

Proof. Indeed, let us consider the region $R = \{(u, v), v \leq \theta_0(u)\}$. If a point P belongs to R, then $f(P)$ also belongs to this region. Indeed, we compute that

$$\begin{aligned} &\frac{v+1}{u+2} - \left(1 - \frac{u}{(u+2)\log 2} \log \frac{2u+2}{u}\right) \\ &\quad = \frac{1}{(u+2)\log 2}\left((v-1)\log 2 + u \log \frac{u+1}{u}\right) \\ &\quad \leq \frac{1}{(u+2)\log 2}\left(-u \log \frac{u+1}{u} + u \log \frac{u+1}{u}\right) = 0. \end{aligned}$$

This proof also shows that the graph of θ_0 is invariant under f. It is easy to show that this graph is above $\mathcal{C}$ when $u \in [1/2, 1]$. The other parts of this graph are obtained by applying f, and since the region upper to $\mathcal{C}$ is stable under f, the graph of θ_0 remains there. □

Remark 2.9. When P belongs to this region then $g(P)$ also belongs to it. Indeed, we have

$$\frac{u}{u+2} - \left(1 - \frac{v+1}{(u+2)\log 2}\log\frac{v+u+3}{v+1}\right)$$
$$= \frac{1}{(u+2)\log 2}\left(-2 + (v+1)\log\frac{v+u+3}{v+1}\right).$$

The function $w \mapsto w\log(1+U/w)$ is non-decreasing when $w \in [1,2]$ and $U \in [2,3]$ (its derivative is $(\log z) + \frac{1}{z} - 1$ for $z = (w+U)/w \in [2,4]$). We thus only have to prove our assertion when $v = \theta_0(u)$ and a simple plot is enough.

Remark 2.10. We note that upon choosing $x = 1/(2^{k+1}-1)$ for a positive integer k, we get the exponent pair

$$(x, \theta_0(x)) = \left(\frac{1}{2^{k+1}-1}, \frac{2^{k+1}-k-2}{2^{k+1}-1}\right),$$

which is $f^k(1,0)$, see page 60 of Graham-Kolesnik [6]. The actual definition of θ_0 was extrapolated from this formula.

Proof of Theorem 1.2. Translating Lemma 2.8 in terms of (κ, λ) and the definition of exponent pairs are all that is required to complete this proof. We note that $\vartheta_0(\kappa) = (\theta_0(2\kappa)+1)/2$.

3. Convex Hull

Given a symmetric subset $S \subset [0,1]^2$ that contains $(0,1)$, we consider the smallest closed convex set $\mathscr{C}(S)$ that contains all the images of S under f and g. Since our set contains $(0,1)$ it contains $(1,0)$ and $(1/3,1/3)$. These are the vertices of the image of $[0,1]^2$ by f and g. On iterating, we find all the (opposite) vertices of the small parallepipedes that we used to build $\mathcal{C}$, from which we conclude that the convex hull of $\mathcal{C}$ is indeed in $\mathscr{C}(\{(0,1)\})$ which we denote by $\mathscr{D}$. Note that, since f and g preserve segments, it is enough to first iterate f and g and, in a second step, to take the convex hull of the final set.

As we see in Figure 2, the curve $\mathcal{C}$ has a singular point at $(\frac{1}{3}, \frac{1}{3})$. But since any other location on the curve in a smooth image of the full curve, the set of points where this phenomenom occurs is in fact dense on $\mathcal{C}$.

Let us describe an algorithmical way of computing $\mathscr{D}$. We start from $(0,1)$ and $(1,0)$, apply f and g, get the convex hull and repeat on the set of vertices obtained. Here is a plot of the first three steps.

At this level, the point $(\frac{1}{3}, \frac{1}{3})$ is becoming useless, and in later steps, it will even become an interior point. So have reached the points

$$(0,1), (1/15, 11/15), (1/7, 4/7), (2/9, 4/9), (4/9, 2/9),$$
$$(4/7, 1/7), (11/15, 1/15), (1, 0)$$

Here is the situation when we reached the step 6, and which shows that finding a pattern to determine which points to keep and which to discard may be intricate.

4. Adding the Huxley and Watt Point

In [1], E. Bombieri and H. Iwaniec improved the Lindelöf exponent beyond what was accessible through the exponent pair method. Their work was extended to yield an exponent pair by M. N. Huxley and N. Watt in [7] and by N. Watt in [13]. This gives us the two (u,v)-points

$$\left(\frac{9}{28}, \frac{9}{28}\right), \quad \left(\frac{89}{280}, \frac{89}{280}\right). \tag{10}$$

We should thus consider $\mathscr{D}^* = \mathscr{C}(\{(0,1), (\frac{89}{280}, \frac{89}{280})\})$. We find that

$$f\left(\left(\frac{89}{280}, \frac{89}{280}\right)\right) = \left(\frac{89}{649}, \frac{369}{649}\right),$$
$$g\left(\left(\frac{89}{280}, \frac{89}{280}\right)\right) = \left(\frac{369}{649}, \frac{89}{649}\right).$$

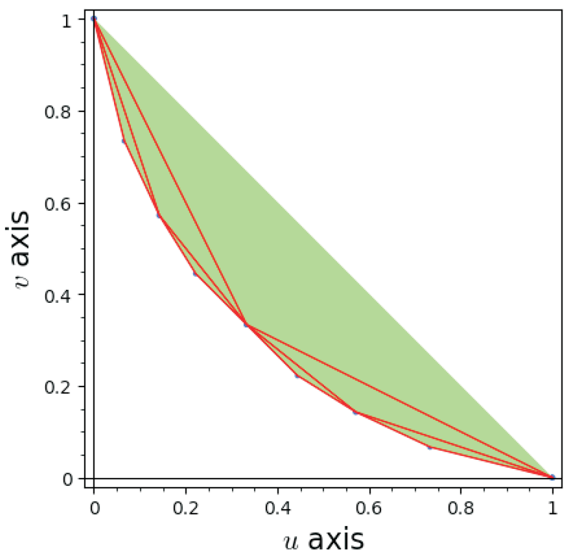

Figure 6. Approximate domain $\mathscr{D}$ with words of length 3, drawing obtained via `ExpPairs.sage/plotDomainC(1)+plotDomainC(2)+plotDomainC(3)`.

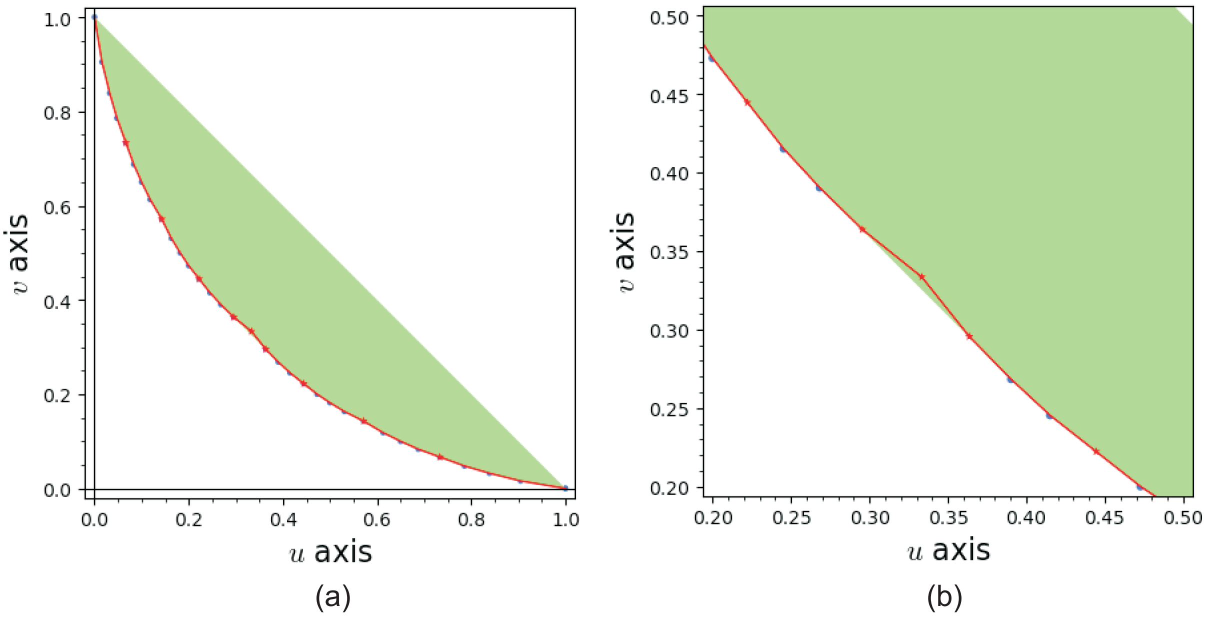

Figure 7. Approximate Domain $\mathscr{D}$. In blue round shape, the points of $\mathcal{C}$ that are on the border, and in red star shape, the ones that are not needed anymore; zoom in (b) on the central part, drawing obtained via `ExpPairs.sage/plotDomainC(5, False)` and adding `.show(xmin=0.2,xmax=0.5,ymin=0.2,ymax=0.5, aspect_ratio=1)` for the second one.

Since the points given in (10) are below the curve $\mathcal{C}$, further points obtained by applying f and or g still stays there. Moreover, as we saw previously, a start like $ffgfg$ localizes the image.

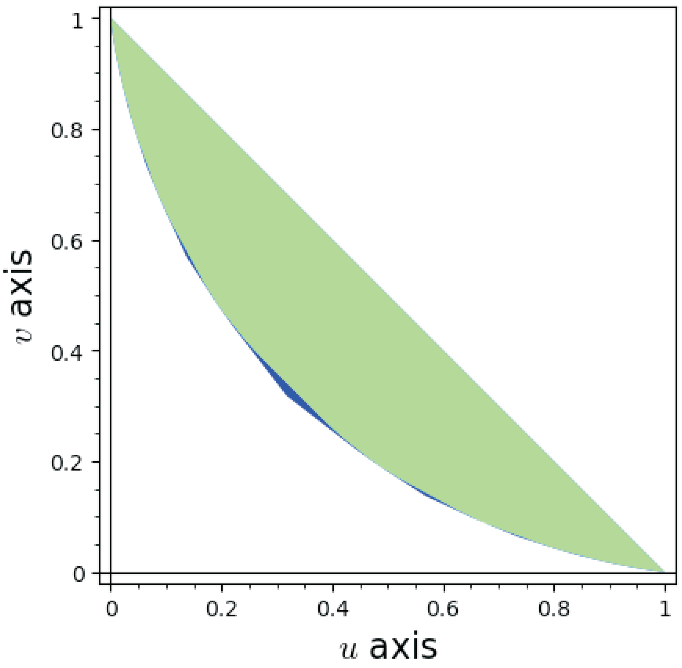

Figure 8. Domain $\mathscr{D}^*$. In blue, the part added by Watt's exponent pair, drawing obtained through `ExpPairs.sage/plotSimpleDomainCS([[0,1], [89/280,89/280]],12, 6, 'blue') + plotSimpleDomainCS([[0,1]], 12)`.

The next figure (in (κ, λ)) shows that the adequation of our model θ_0 to this case is not as good as before but still within an acceptable magin.

The question then is to find some equivalent form to Theorem 1.2. The function θ_0 (resp. ϑ_0 if we express it in the variables (κ, λ)) has (rather strickingly) a simple form, but we may as well replace it by θ^* (resp. ϑ^*) which parametrizes the (lower) border of $\mathscr{D}^*$.

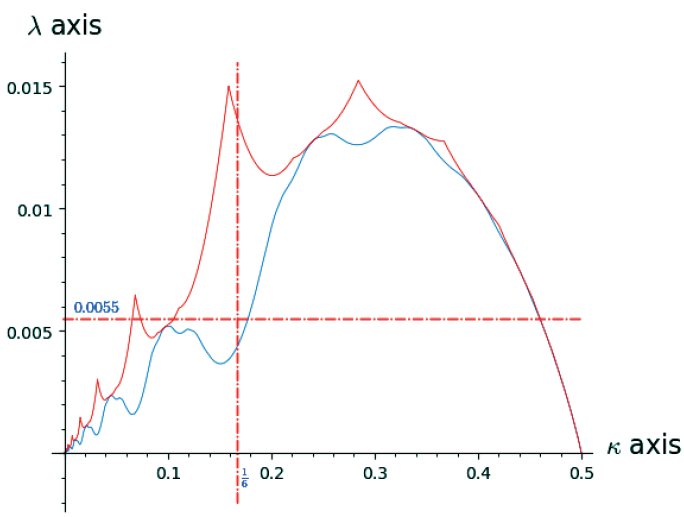

Figure 9. Difference between ϑ_0 and the optimal exponent pair from the Watt (resp. van der Corput) domain in red [top] (resp. in blue [below]), drawing obtained via `ExpPairs.sage/compareModelConvHull(8)+compareModelConvHull(8)`.

Theorem 4.1. *The θ^* be the continuous decreasing convex function on $[0,1]$ that parametrizes the lower border of $\mathscr{D}^*$ and ϑ^* be the continuous decreasing convex function on $[0,1/2]$ defined by $\vartheta^*(\kappa) = (2\theta^*(2\kappa)+1)/2$. Let S be an exponential sum of monomial type and parameters N and F. Then, for every $\kappa \in \left[0,\frac{1}{2}\right]$ and every $\varepsilon > 0$, we have $S \ll_\varepsilon F^{\kappa+\varepsilon}N^{\vartheta^*(\kappa)+\varepsilon}$. The functional equation $\theta^*\circ\theta^* = \theta^*$ holds, as a consequence of the symmetry $(u,v)\mapsto(v,u)$.*

An upper bound for θ^* is provided by the step-function θ_1 (or ϑ_1 in the variables (κ,λ)) that links the following points:

$$[0,1], \left[\frac{89}{1387},\frac{1018}{1387}\right], \left[\frac{89}{649},\frac{369}{649}\right], \left[\frac{369}{1667},\frac{738}{1667}\right],$$
$$\left[\frac{89}{280},\frac{89}{280}\right], \left[\frac{738}{1667},\frac{369}{1667}\right], \left[\frac{369}{649},\frac{89}{649}\right],$$
$$\left[\frac{1018}{1387},\frac{89}{1387}\right], [1,0].$$

This leads to a rather decent approximation of our border, as shown by the next plots.

This is obtained via

```
BasePlot = plotSimpleDomainCS
           ([[0,1], [89/280,89/280]],
           10, 6, 'blue')
BasePlot += plotSimpleDomainCS([[0,1]], 10)
BasePlot += list_plot(rationalCHullW(2),
            zorder=3, color='brown',
            plotjoined = True, figsize = 6)
BasePlot.show()
```

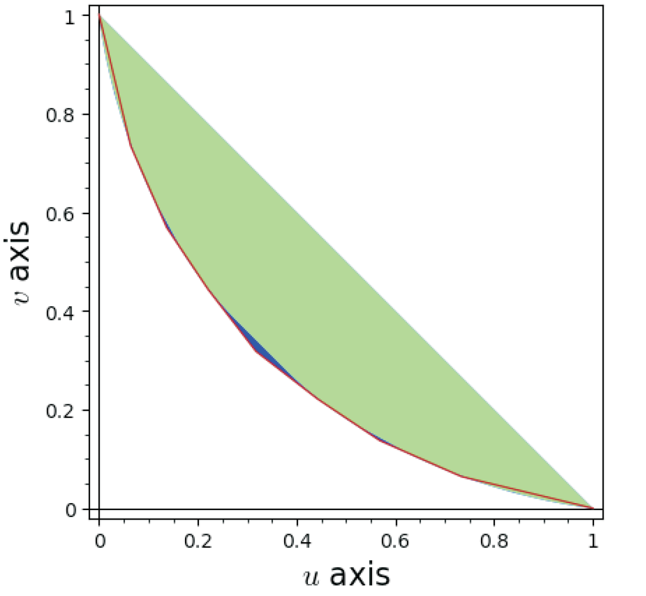

Figure 10. The Watt (resp. van der Corput) domain in blue (resp. in green) and the rational upper bound in brown.

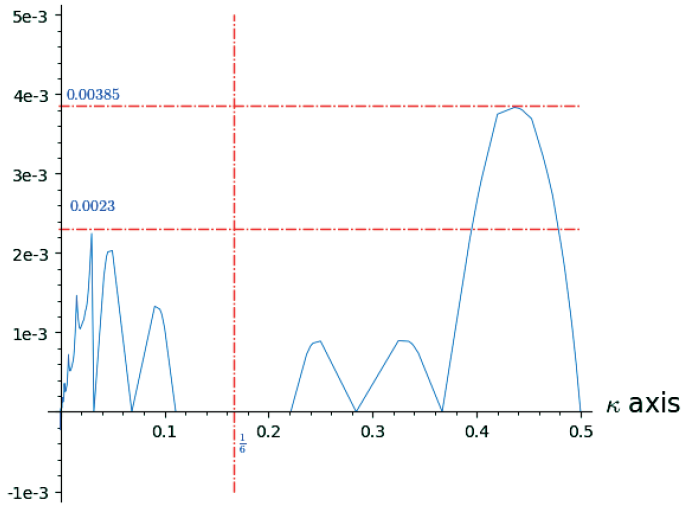

Figure 11. Difference $\min(\vartheta_0,\vartheta_1)$ and the optimal exponent pair from the Watt domain, drawing obtained via `ExpPairs.sage/compareFiniteModelConvHull(8)`

5. A Remark Concerning *p*-dimensional Exponent Pairs

In [10], B. R. Srinivasan developped a multi-dimensional theory of exponent pairs. Roughly speaking, let $\varphi(x_1,x_2,\ldots,x_p)$ be a smooth 'polynomial-like' function whose partial derivative with respect to x_i remains of size F_i when x_i is about N_i. Then a pair $(\kappa,\lambda) \in \left[0,\frac{1}{2(p+1)}\right]\times\left[\frac{2p-1}{2p},1\right]$ is said to be a p-dimensional exponent pair when, given the above data and $\varepsilon>0$, we have

$$\sum_{\forall i, n_i\sim N_i} e^{2i\pi\varphi(n_1,\ldots,n_p)} \ll_\varepsilon \left(\prod_i F_i\right)^{\kappa+\varepsilon}\left(\prod_i N_i\right)^{\lambda+\varepsilon}.$$

B. R. Srinivasan continues by showing that one can form two other p-dimensional exponent pairs from a given one, say (κ,λ), by the expressions

$$\left(\frac{\kappa}{2(1+p\kappa)},\frac{(2p-1)\kappa+\lambda+1}{2(1+p\kappa)}\right), \quad \left(\lambda-\frac{1}{2},\kappa+\frac{1}{2}\right).$$

The change of variables

$$u = 2p\kappa, \quad v = 2p\lambda - 2p + 1$$

leads to the rules (6), and the geometrical problem is thus unchanged! We then infer the next result from Theorem 1.2 and Theorem 4.1.

Theorem 5.1. *For any* $\kappa \in \left[0, \frac{1}{2(p+1)}\right]$, *the couple* $\left(\kappa, \frac{\vartheta_0(p\kappa)+p-1}{p}\right)$ *is a p-dimensional exponent pair. The same is true of* $\left(\kappa, \frac{\vartheta^*(p\kappa)+p-1}{p}\right)$, *where* ϑ^* *is defined in Theorem 4.1.*

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Mathematics and the Machines

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Abstract. This article describes some instances of the use of computers and computer techniques in proving some landmark results in Mathematics. Recent activities in this direction, namely, methods involving Artificial Intelligence which try to imitate human intelligence and intuition are also discussed. The style followed in this article is that of a story telling to young graduate students.

1. Introduction

It is well established by now that computer calculations have not only helped in verifying Mathematical results but they have also acted as a significant aid in creating Mathematics. Many Mathematicians have used this facility to produce significant results. However, with the progress of time, this facility has advanced with many more useful features. The catch terms nowadays are *Artificial intelligence, Big Data, Deep Learning* etc.

A common approach in the use of computers has generally been to compute invariants, observe patterns or find some significant numerical values corresponding to large inputs which are usually hard to compute by hand. This may, for example, involve computing determinants of matrices of large size, computing invariants such as polynomials or groups or homology, etc., and sometimes (if not very often) observing patterns such as the behavior of a particular feature over a certain range. Based on the patterns observed we make deductions and conjectures which are explored further.

To use this facility to our benefit to search for patterns, we need to use Data. We use computers to generate data and also to observe patterns in data. In this article, we give an overview of tools accessible and briefly discuss how have these tools grown up over time to help Mathematicians obtain directions and leads in proving results which have thus far remained unanswered. We present some examples of such instances which have shaped Mathematics significantly.

2. The Four Color Problem

This was probably the first instance where a Mathematical problem could be completely solved with the aid of a computer. The four color problem is simple, however establishing an acceptable proof of this problem took a long time.

To describe the problem, we recall that the atlases which we used in our geography classes during our school contained maps. Map of a country was always drawn along with its neighbouring countries, i.e. those which share a boundary with the country. A coloring of a map is assignment of colors to the countries in the map such that no two countries which share a common boundary receive the same color. The question which came to be known as four color problem was whether four colors would be sufficient to color any map on the plane.

The dual version of this problem is in graph theoretic terms. The graphs which can be drawn on the plane without edges meeting at points others than vertices are called planar graphs. A coloring of a graph is to associate colors to its vertices in such a way that adjacent vertices receive different colors. The problem, thus, was to show that to color a planar graph four distinct colors would be sufficient.

By now it is well known that the first important solution of this problem came as a proof by using computers. Two Mathematicians Appel and Haken [1] worked on this and established that it is possible to color a planar graph with only four colors. Their work was based on the findings of earlier researchers. Appel and Haken eventually showed that a counter-example to the four-color problem did not exist.

If a map was a counter-example to the four-color problem then there would be smallest such map. This was called a minimal counter example to the four-color problem. A certain pattern of countries on a map is termed a configuration. An unavoidable set of configurations would be a set of configurations satisfying certain conditions such that a counter example to four color problem must contain at least one of them. A reducible configuration is a configuration that can not appear in a minimal counter example to the four-color problem.

Appel and Haken were able to find an unavoidable set of reducible configurations which showed that a minimal counter example to the four-color problem can not exist. For this, they had to check about 1500 configurations using computers.

Although this proof involved intensive human effort and manual checking of lists spread over about 400 pages, still the equally important role was that delegated to computer. An account of which can be found in [2] and [3].

3. Interplay between Geometry and Topology: Work of Robert Riley

One of the objects of study in Mathematics has been “Manifolds” (whenever we are interested in understanding structures). These objects are topological spaces that locally look like Euclidean spaces. The well-known examples of these are Euclidean spaces themselves together with other objects occurring commonly and naturally. Depending on their local similarity to Euclidean space the dimension of a manifold is defined. For example, a straight line or a circle would be a 1-dimensional manifold. A Sphere (think of the surface of a football), torus (surface of a doughnut), or piece of a plane would be a two-dimensional manifold. A topological object that locally looks like Euclidean 3 space, the space in which we live, will be a three-dimensional manifold. For example a solid ball and many day-to-day objects around us would fall into this category

Here is an example of a 3-manifold which is not quite familiar to many of us. Take a solid cube. On each of the six faces of the cube write numbers 1, 2, 3, 4, 5 and 6. Take two such identical cubes and glue face numbered “k” of both the cubes together successively. This object is a three dimensional manifold called three dimensional sphere. We would like these manifolds to have some nice features, e.g., we want them to be in one piece (connected), would like them to be closed (so that we do not fall off an edge) and would like them to be "finite" that is to have finite volume. In addition, in favor of precision and to avoid confusion we would like them to be orientable, meaning thereby that when we travel on the manifold starting from one point of the manifold and return to it we would have not changed sides, our left remains left and right remains right side. An important notion is that of simple connectedness. A manifold will be called simply connected if every simple closed curve in the manifold can be shrunk within the manifold to a point. That is if every non self intersecting loop in the manifold can be shrunk to a point while staying in the manifold. Such loops in the manifold help in associating a very useful algebraic object to the manifold called a group (fundamental group). If all loops can be shrunk to a point the group is trivial. The three sphere mentioned above has all these properties.

It was in 1904 that the French Mathematician Poincare asked a question -– whether every simply connected closed 3-manifold was a 3-Sphere? This problem came to be known as the Poincare conjecture. The subject Topology in Mathematics developed around attempts to answer this question.

This lead to further intensive studies involving manifolds and generalizations. One of the natural ways to study and understand Mathematical objects is to simplify them by decomposing them into smaller and simpler familiar pieces. The sub-objects in the case of manifolds are called submanifolds. Thus, if we understand all the submanifolds sitting inside a manifold we will be in a comfortable position to understand the manifold. For example in a 2-sphere we find circles of various radii. Similarly in a torus we find circles of two special types, longitudinal and meridional, which help in understanding the torus. Coming back to the Poincare's question one wonders what can be submanifolds in a three dimensional Sphere?

They can be submanifolds of dimensions one and dimension two. The submanifolds of dimension two would be called surfaces and the submanifolds of dimension one would be called circles. Circles and continuous maps of circles in a three dimensional sphere are called knots. The subject Knot Theory in Mathematics pertains to the study of knots. It turns out that the object which is complement of a knot in a three sphere is an important class of three manifolds and it is crucial to understand these objects. Fundamental group associated to this complement for a given knot is called the group of that knot or the knot group. This group carries important information about the structure of the manifold. Sometimes when the group is not quite familiar we embed it into another familiar group via mappings called homomorphisms and try to deduce results about the group from the known properties of the group in which we have embedded it. One such procedure is know as representation of a group. Representations into permutation groups are interesting from structural point of view.

In early years of his career (1967-–68) Robert Riley [4] studied a paper of R. H. Fox in which Fox had distinguished two knots named as granny and square knots. The techniques used by Fox involved representation of groups of knot into permutation group. More specifically into a permutation group of degree five, denoted A_5. Robert Riley wrote out explicit procedures to find all A_5 representation of a knot group. This work was done using computer programming in Fortran language. Over a time period spread over many months Riley did required computer analysis and was able to find a nice and useful representation of the knot group of figure eight knot. At this point the findings revealed that the purely topological object namely the complement of figure eight knot in three dimensional sphere also had a geometric structure which was hyperbolic. These turned out to be a very significant finding in the study of 3-manifolds.

4. The Geometrisation: Thurston's Work

Around the same time in common room of Mathematics Department of Warwick University William Thurston met Riley. In his writing, [5] Thurston mentioned that Riley's construction of a number of beautiful examples with the aid of computers gave a big impetus to prove the following significant result:

The interior of a compact 3-manifold M with non empty boundary has a hyperbolic structure iff M is prime, homotopically atoroidal and not homeomorphic to quotient of Tours $\times[0, 1]$ by $\mathbb{Z}_2$.

Now here comes the role of surfaces in 3-manifolds. A 3-manifold being prime would roughly mean that there are no real 2-spheres in it. Homotopically atoroidal would similarly, roughly, mean that a torus sitting inside the 3-manifold can be pushed to lie in the boundary of the manifold. This also led Thurston to propose the result – A knot complement has a geometric structure if and only if the knot is not a satellite knot. Motivated by this finding Thurston posed the following problem: Interior of every compact 3-manifold has a canonical decomposition into pieces that have geometric structures.

The decomposition stated above is in two steps. The first step (called the prime decomposition) which is obtained by repeatedly cutting the 3-manifold along embedded 2-spheres and then capping the void so created by 3-balls so as to obtain simpler manifolds. Following this the second step involves cutting the 3-manifold along embedded tori (called JSJ decomposition) to obtain manifolds whose boundaries are tori. It is a result of Kneser and Milnor that the process of cutting a 3-manifold along 2-spheres terminates after a finite number of steps and the resulting prime manifolds are unique up to homeomorphisms.

By using a computer program, Thurston found patterns that gave rise to hyperbolic 3-manifolds and the patterns which could not be shown to have hyperbolic structures actually had other geometries. This came to be known as the geometrization conjecture which was proved to be true by the works of Richard Hamilton and Grigory Perelman, which also included setting of the poincare conjecture.

5. Advanced Techniques: Machine Learning

With the advancement of the field of computer science and methods of computations, powerful techniques of computations have evolved. These techniques try to imitate human intelligence and hence are also appropriately termed as techniques of artificial intelligence. This involves studies of patterns in observations or surverys made and deductions based on recognizing these patterns and interpretating them. All this work is to be carried out by machines by making the machines learn techniques to solve problems. We humans supervise these learning and sometimes these learnings become unsupervised when machine learns to answer the questions posed to it.

Thus if we want to obtain a particular result which is suspected to be true then we generate data which can be used to train the machine to find patterns and make deductions. We may then use these outcomes of our study to propose a conjecture or eventually prove theorem. Based on the paper [6] we present here two instances of use of these advanced techniques to make important deductions which are significant progresses in the subject of study.

The first of these instances is in the field of knot theory. In this case, it was thought that there exists undiscovered relation among hyperbolic and algebraic invariants of knots. The steps mentioned above were performed and a pattern was detected. The pattern was among a set of geometric invariants and an invariant based on algebra called signature. It was observed that the algebraic quantity signature was being controlled by cusp geometry. This study gave an idea to investigate these invariants and eventually leading to establishing a relation between signature, "slope" and volume.

The second instance is in the field of representation theory which was also mentioned in a section above. A serious thought on the approach mentioned previously may lead us to understand that in representation theory we would be concerned about symmetry and when the group consists of linear transformations the symmetry being considered is linear symmetry. The basic objects of study in this area are what are known as irreducible representations. In many cases there is a polynomial associated to these representations called the Kazhdan Lustzig (or briefly KL) polynomials. Computing KL polynomials has been an important task and it turned out that these polynomials can be computed from a graph associated to them called the Bruhat interval. However, these graphs are of very large size when the KL polynomials are interesting.

Computing KL polynomial from its Bruhat interval was taken as the main task in this work. After performing the above stated procedures it was found that a subgraph of Bruhat interval was sufficient to compute the KL polynomial. The graph was further investigated and a decomposition of the graph was obtained. It turned out that the KL polynomial can in fact be calculated from the decomposition.

6. Ramanujan Machine

A group of researchers [7] have proposed to use algorithms to devise formulas that give rise to constants such as e and π and reveal some underlying Mathematical structure. They have named this approach Ramanujam Machine. Using this they have been successful in generating numerous constants and proposing conjectures some of which have been proven to be true and some are yet to be settled. Unlike automated theorem proving and based on numerical observations, their method proposes conjectures without prior information of any underlying mathematical foundation. They seem to propose to replace Mathematical intuition which had been the basis of discoveries made by Ramanujan.

Conclusion

Thus it is clear that the machine has played significant role in verifying and creating mathematics. It is going to remain an indispensable tool for working mathematians.

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Starlike and Convex Univalent Functions

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Dedicated to Professor Om P. Ahuja

Abstract. An analytic function defined on the open unit disk in the complex plane is univalent if it is one-to-one there. The ratio of nth coefficient to second coefficient of univalent functions is bounded by n. This was conjectured by Bieberbach in 1916 and this was proved only in 1984–85. The conjecture motivated many developments in univalent function theory. One of them is the study of subclasses of univalent functions having certain geometric properties. In this article, we shall be interested in univalent functions whose range is either a convex domain or a starlike domain (a domain in which the line segment joining the origin to any point in it lies inside it).

Keywords. Growth, Distortion, Fixed second coefficient, Radius, Subordination, Starlike, Convex, Functions with positive real part.

2020 Mathematics Subject Classification: 30C45, 30C55.

1. Introduction

Riemann mapping theorem asserts that every proper simply connected domain G is conformally equivalent to the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and hence the properties of a function defined on a proper simply connected domain can be obtained via the Riemann mapping function

$g : G \to \mathbb{D}$ from the corresponding function defined on $\mathbb{D}$. We shall be interested in the inverse of the Riemann mapping functions; these are functions $f : \mathbb{D} \to \mathbb{C}$ that are univalent (=one-to-one analytic). In view of this, the study of univalent functions is restricted to the function defined on $\mathbb{D}$. As an application of the open mapping theorem for an analytic function, we can show that the derivative of univalent functions is non-vanishing and the univalence is not affected by the translation and dilation. Therefore, we focus on the class $\mathcal{A}$ of analytic functions $f : \mathbb{D} \to \mathbb{C}$ normalized by $f(0) = 0$ and $f'(0) = 1$. Let $\mathcal{S}$ be its subclass consisting of univalent functions. The famous Bieberbach-de Brange's theorem [10] states that if $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{S}$, then $|a_n| \leqslant n$ with equality if and only if f is a rotation of the Koebe function $\kappa(z) = z/(1-z)^2$. This theorem for $n = 2$, known as Bieberbach theorem [7], gives the growth and distortion theorems for functions in the class $\mathcal{S}$. The famous Koebe one-quarter theorem that $\{w \in \mathbb{C} : |w| < 1/4\} \subseteq f(\mathbb{D})$ for each $f \in \mathcal{S}$ also follows from the Bieberbach theorem. To see this, note that if $f \in \mathcal{S}$ omits a value w, then the second coefficient of the function $wf/(w - f) \in \mathcal{S}$ satisfies

$$\left| a_2 + \frac{1}{w} \right| \leqslant 2$$

which leads to $|w| \geqslant (2 + |a_2|)^{-1}$ and, hence, $|w| \geqslant 1/4$. By an application of the Bieberbach theorem to a suitable composition of $f \in \mathcal{S}$ with a Möbius transform, it can be shown that

$$\left| \frac{zf''(z)}{f'(z)} - \frac{2r^2}{1-r^2} \right| \leqslant \frac{4r}{1-r^2}, \quad |z| = r < 1. \tag{1.1}$$

The above inequality is useful in obtaining the bounds for $|f'(z)|$ (distortion), $|f(z)|$ (growth) and the radius of convexity of the class $\mathcal{S}$. Indeed, the inequality (1.1) implies the following growth and distortion results for the class $\mathcal{S}$:

$$\frac{1-r}{(1+r)^3} \leqslant |f'(z)| \leqslant \frac{1+r}{(1-r)^3}, \quad |z| = r < 1$$

and

$$\frac{r}{(1+r)^2} \leqslant |f(z)| \leqslant \frac{r}{(1-r)^2}, \quad |z| = r < 1.$$

This growth inequality shows that the class $\mathcal{S}$ is locally bounded and hence a normal family. It also follows that $\mathcal{S}$ is compact. In view of the importance of the estimate for the coefficients of univalent functions, several researchers have studied the coefficient problems for functions with certain specific geometric properties.

In this article, we shall be interested in two such geometric properties, namely convexity and starlikeness of univalent functions. These are functions whose image domains are either convex or starlike with respect to the origin. Though the classes are defined by geometric conditions, an analytic description of each of these functions will be given. Using this we obtain several properties like growth and distortion estimates. Several results related to the coefficients, inclusion, radius are also included.

For information on univalent functions, see Duren [11]. For starlike and convex functions and several other related functions, see the books by Goodman [16,17] and Thomas et al. [60]. For starlike and convex functions of several variables, see Gong [15]. For general theory of convolutions and many application, see the monograph by Ruschewey [49]. For subordination related results, we recommend the monograph by Miller and Mocanu [30].

2. Starlike Functions

A line segment joining the points z_1 and z_2 is the set

$$[z_1, z_2] := \{tz_1 + (1-t)z_2 : 0 \leqslant t \leqslant 1\}.$$

A domain G is starlike with respect to a point $z_0 \in G$ if the line segment $[z, z_0]$ joining the points z and z_0 lies in G for every $z \in G$. A function $f : \mathbb{D} \to \mathbb{C}$ whose image $f(\mathbb{D})$ is starlike with respect to the origin is called a starlike function. The class of all starlike functions f in $\mathcal{S}$ is denoted by $\mathcal{ST}$. Since the points on the line joining z to the origin are tz where $0 \leqslant t \leqslant 1$, a function f is in the class $\mathcal{ST}$ if and only if $f(tz) \in f(\mathbb{D})$ for all t with $0 \leqslant t \leqslant 1$ and for all $z \in \mathbb{D}$. A region G is **convex** if the line segment $[z_1, z_2]$ joining z_1 and z_2 lies entirely in G. A function $f \in \mathcal{S}$ is convex if $f(\mathbb{D})$ is convex and the class of all convex functions $f \in \mathcal{S}$ is denoted by $\mathcal{CV}$. By definition, both the classes $\mathcal{CV}$ and $\mathcal{ST}$ are subclasses of $\mathcal{S}$. Geometrically, it is clear that every convex function is starlike and so $\mathcal{CV} \subset \mathcal{ST} \subset \mathcal{S}$.

Example 2.1 Koebe function. For the Koebe function $\kappa : \mathbb{D} \to \mathbb{C}$ defined by $\kappa(z) = z/(1-z)^2$, we have $\kappa(\mathbb{D}) = \mathbb{C}\backslash(-\infty, -1/4]$, the entire complex plane except for a slit along the negative real axis from $w = -\infty$ to $w = -\frac{1}{4}$, and it is clearly starlike with respect to the origin. Thus, the Koebe function κ belongs to the class $\mathcal{ST}$ of starlike functions.

The following theorem gives growth, distortion and covering estimates for the class of starlike functions. These inequalities are true for all functions in the class $\mathcal{S}$. Since $\mathcal{ST} \subset \mathcal{S}$, the results follow. The sharpness follows as the Koebe function κ is in the class $\mathcal{ST}$.

Theorem 2.1. *If f is in the class $\mathcal{ST}$, then*

$$\frac{r}{(1+r)^2} \leqslant |f(z)| \leqslant \frac{r}{(1-r)^2},$$

$$\frac{(1-r)}{(1+r)^3} \leqslant |f'(z)| \leqslant \frac{(1+r)}{(1-r)^3},$$

and for each $k \geqslant 2$

$$|f^k(z)| \leqslant \frac{k!(k+r)}{(1-r)^{k+2}}.$$

In addition, we have $\mathbb{D}_{1/4} \subset f(\mathbb{D})$. All of these results are sharp, with equality if and only if f is some rotation of the Koebe function κ. Here $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$.

Other properties of starlike and convex functions are obtained by characterizing these functions by using the functions with positive real part. To give an analytic characterization of starlike functions, we need the following theorem.

Theorem 2.2 Hereditary property of starlike functions. *A function f in $\mathcal{A}$ is in the class $\mathcal{ST}$ if and only if $f(\mathbb{D}_r)$ is starlike with respect to the origin for each r with $0 < r < 1$. Equivalently, a function f is in the class $\mathcal{ST}$ if and only if the function $f_r : \mathbb{D} \to \mathbb{C}$ defined by $f_r(z) = r^{-1}f(rz)$ belongs to the class $\mathcal{ST}$ for each r with $0 < r < 1$.*

Proof. Suppose that f is a starlike function. Let $t \in (0,1)$ be fixed and let $z \in \mathbb{D}$. Since $f(\mathbb{D})$ is starlike with respect to the origin, we have $tf(z) \in f(\mathbb{D})$ and hence $f^{-1}(tf(z)) \in \mathbb{D}$. Let $w : \mathbb{D} \to \mathbb{C}$ be defined by $w(z) = f^{-1}(tf(z))$ so that $tf(z) = f(w(z))$. Using this, we see that the function $f_r : \mathbb{D} \to \mathbb{C}$ defined by $f_r(z) = r^{-1}f(rz)$ satisfies

$$\begin{aligned} tf_r(z) &= r^{-1}tf(rz) = r^{-1}f(w(rz)) \\ &= r^{-1}f(rw_1(z)) = f_r(w_1(z)) \end{aligned}$$

where the function $w_1 : \mathbb{D} \to \mathbb{C}$ is defined by $w_1(z) = w(rz)/r$. Since f is univalent, it follows that w is a single-valued analytic function that maps $\mathbb{D}$ into $\mathbb{D}$ and

$$w(0) = f^{-1}(tf(0)) = f^{-1}(0) = 0.$$

Therefore, by the Schwarz lemma, $|w(z)| \leqslant |z|$ for each $z \in \mathbb{D}$. The function w_1 satisfies

$$|w_1(z)| = \frac{|w(rz)|}{r} \leqslant \frac{|rz|}{r} = |z|.$$

Therefore, $tf_r(z) = f_r(w_1(z)) \in f_r(\mathbb{D})$, proving that f_r is in the class $\mathcal{ST}$.

To prove the converse, note that the starlikeness of f_r implies the starlikeness of rf_r. Since

$$f(\mathbb{D}) = \bigcup_{0<r<1} rf_r(\mathbb{D})$$

and the union of domains starlike with respect to the origin is again a domain starlike with respect to the origin, the result now follows. □

We define the class $\mathcal{P}$ of Carathéodory functions (or functions with positive real part) to be the class of all analytic functions $p : \mathbb{D} \to \mathbb{C}$ with $p(0) = 1$ satisfying $\operatorname{Re} p(z) > 0$. Starlike functions can be characterized analytically by relating it to a function in the class $\mathcal{P}$. The class $\mathcal{P}$ is well-studied and several properties of them are well-known. For example, if $p \in \mathcal{P}$ and $p(z) = 1+c_1z+c_2z^2+\cdots$, then $|c_n| \leqslant 2$ and equality for $p(z) = (1+z)/(1-z)$. We also have the following theorem.

Theorem 2.3. *If the function p is in the class $\mathcal{P}$ and $p(z) = 1 + c_1z + c_2z^2 + \cdots$, then*

$$|c_2 - \mu c_1^2| \leqslant 2\max\{1, |2\mu - 1|\}, \quad \mu \in \mathbb{C}. \tag{2.1}$$

For $0 < \mu < 1$, the following improved estimates hold:

$$|c_2 - \mu c_1^2| + \mu|c_1|^2 \leqslant 2 \quad (0 < \mu \leqslant 1/2) \tag{2.2}$$

and

$$|c_2 - \mu c_1^2| + (1-\mu)|c_1|^2 \leqslant 2 \quad (1/2 \leqslant \mu < 1). \tag{2.3}$$

Let the functions $p_0, p_1 \in \mathcal{P}$ be given by

$$p_1(z) = \frac{1+z}{1-z}, \quad \textit{and} \quad p_2(z) = \frac{1+z^2}{1-z^2}.$$

The inequality (2.1) is sharp for the function p_1 when $|2\mu - 1| > 1$ and for function p_2 when $|2\mu - 1| \leqslant 1$. The inequalities (2.2) and (2.3) are sharp for both p_1 and p_2.

The following theorem is an application of the hereditary property of starlike functions.

Theorem 2.4. *A function f in $\mathcal{A}$ is in the class $\mathcal{ST}$ if and only if*

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > 0 \quad (z \in \mathbb{D}). \tag{2.4}$$

Equivalently, the function f is in the class $\mathcal{ST}$ if and only if the function $p : \mathbb{D} \to \mathbb{C}$ defined by

$$p(z) = \frac{zf'(z)}{f(z)} \tag{2.5}$$

belongs to the class $\mathcal{P}$.

Proof. Let $f \in \mathcal{ST}$. By the hereditary property of starlike functions (Theorem 2.2), the function rf_r is starlike for each r with $0 < r < 1$. Thus, the function f maps $\mathbb{D}_r$ into a domain starlike with respect to the origin. Therefore, the image curve C_r of the circle $|z| = r$ under the function f bounds a starlike domain. Geometrically, this means that $\arg(f(z))$ increases as z moves around the circle $|z| = r$. Thus, we have

$$\frac{\partial}{\partial\theta}\arg(f(re^{i\theta})) \geqslant 0.$$

Recall that $\log z = \log|z| + i\arg z$ and so $\arg z = \operatorname{Im}\log z$. By using this, with $z = re^{i\theta}$, we have

$$0 \leqslant \frac{\partial}{\partial\theta}\arg(f(re^{i\theta})) = \operatorname{Re}\frac{zf'(z)}{f(z)}.$$

The function $p : \mathbb{D} \to \mathbb{C}$ defined by (2.5) is clearly well-defined as univalence of f shows that $f(z) \neq 0$ for $z \neq 0$. At $z = 0$, the function p has a removable singularity and we remove it by defining $p(0) = 1$. It is easy to see that p is analytic and therefore, by minimum principle, we have $\operatorname{Re} p(z) > 0$ and (2.4) holds. This also proves p as defined by (2.5) is in the class $\mathcal{P}$.

To prove the converse, assume that $f \in \mathcal{A}$ satisfies (2.4). Since $f \in \mathcal{A}$, we have $f(0) = 0$. We claim that $f(z) \neq 0$ for $z \neq 0$. If $f(z_0) = 0$ for some $z_0 \neq 0$, then z_0 is a pole of the function p defined by (2.5) contradicting the analyticity of p. Therefore, the function f has only one zero in $\mathbb{D}$. As before let C_r be the image of the circle $|z| = r$ under f. Then, by argument principle, we see that the winding number $n(C_r; 0)$ of the circle C_r around the origin is given by

$$n(C_r; 0) = \frac{1}{2\pi i}\int_{C_r}\frac{dw}{w} = \frac{1}{2\pi i}\int_{\partial\mathbb{D}_r}\frac{f'(z)}{f(z)}dz = 1.$$

Therefore, the curve C_r winds around the origin exactly once. But, by using (2.4), we see that $\arg(f(z))$ increases as z moves around the circle $|z| = r$ and therefore there is no self-intersection in $f(C_r)$. In other words, the function f is one-to-one on $\partial\mathbb{D}_r$ for each r with $0 < r < 1$. By the Darboux Theorem, we see that f is univalent in $\mathbb{D}_r$ and hence univalent in $\mathbb{D}$. Since $\arg(f(z))$ increases as z moves around the circle $|z| = r$, it follows that the domain $f(\mathbb{D}_r)$ bounded by C_r is starlike with respect to the origin. Therefore, $f(\mathbb{D})$ is also starlike with respect to the origin. □

Corollary 2.1. *The class $\mathcal{ST}$ is a compact normal family.*

Proof. Since $\mathcal{ST} \subset \mathcal{S} \subset \mathcal{H}(\mathbb{D})$, we see that the class $\mathcal{ST}$ is normal as it is a subset of the normal family $\mathcal{S}$. To prove the compactness, consider a sequence $\langle f_n \rangle$ of functions $f_n \in \mathcal{ST}$. By the compactness of $\mathcal{S}$, there is a subsequence $\langle f_{n_k} \rangle$ of $\langle f_n \rangle$ converging to some $f \in \mathcal{S}$. Since $f_{n_k} \in \mathcal{ST}$, we have

$$\operatorname{Re}\left(\frac{zf'_{n_k}(z)}{f_{n_k}(z)}\right) > 0.$$

Weierstrass' Theorem shows that $f'_{n_k} \to f'$ as $k \to \infty$. By letting $k \to \infty$, we get

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \geqslant 0.$$

By minimum principle, we obtain $\operatorname{Re}(zf'(z)/f(z)) > 0$ and therefore the function f is in the class $\mathcal{ST}$. This proves the compactness of the class $\mathcal{ST}$. □

Theorem 2.4 is very useful in generating starlike functions.

Example 2.2. The Koebe function $\kappa : \mathbb{D} \to \mathbb{C}$ defined by $\kappa(z) = z/(1-z)^2$ satisfies

$$\operatorname{Re}\frac{z\kappa'(z)}{\kappa(z)} = 1 + \operatorname{Re}\frac{2z}{1-z} = \operatorname{Re}\frac{1+z}{1-z} > 0,$$

and therefore, by Theorem 2.4, the Koebe function κ is in the class $\mathcal{ST}$.

Example 2.3. If $f \in \mathcal{A}$ satisfies the condition

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1$$

then f is in the class $\mathcal{ST}$. The converse of this is not true in general as the Koebe function shows.

The given condition says that $w = zf'(z)/f(z)$ lies in the disk $\{w : |w - 1| < 1\}$ and this disk is clearly contained in the right-half plane. Indeed, using $\operatorname{Re} w \geqslant -|w|$, we see that

$$\operatorname{Re}\frac{zf'(z)}{f(z)} = 1 + \operatorname{Re}\left(\frac{zf'(z)}{f(z)} - 1\right) \geqslant 1 - \left|\frac{zf'(z)}{f(z)} - 1\right| > 0$$

and, therefore, by Theorem 2.4, the function f is in the class $\mathcal{ST}$. In particular, the function f_0 defined by $f_0(z) = ze^z$ satisfies

$$\frac{zf_0'(z)}{f_0(z)} = 1 + z \quad \text{and hence} \quad \left|\frac{zf_0'(z)}{f_0(z)} - 1\right| < |z| < 1.$$

Thus, the function f_0 is in the class $\mathcal{ST}$. This could be seen directly also as

$$\operatorname{Re}\frac{zf_0'(z)}{f_0(z)} = 1 + \operatorname{Re} z \geqslant 1 - |z| > 0.$$

Example 2.4. If $f \in \mathcal{A}$ with $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ satisfies

$$\sum_{k=2}^{\infty} k|a_k| \leqslant 1,$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1$$

and hence f is in the class $\mathcal{ST}$.

It is not difficult to see that the function f is in the class $\mathcal{S}$ and so $f(z) \neq 0$ for $z \neq 0$. Also

$$\begin{aligned} &|zf'(z) - f(z)| - |f(z)| \\ &= \left|\sum_{k=2}^{\infty}(k-1)a_k z^k\right| - \left|z + \sum_{k=2}^{\infty} a_k z^k\right| \\ &\leqq \sum_{k=2}^{\infty}(k-1)|a_k||z|^k - \left(|z| - \sum_{k=2}^{\infty}|a_k||z|^k\right) \\ &\leqq |z|\left(\sum_{k=2}^{\infty} k|a_k| - 1\right) \leqq 0 \end{aligned}$$

Therefore, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leqslant 1$$

and hence, by Maximum Modulus Theorem, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1.$$

By Example 2.3, the function f is in the class $\mathcal{ST}$.

A domain G is said to be k-fold symmetric if a rotation of G about the origin through an angle $2\pi/k$ carries G onto itself. A function $f : \mathbb{D} \to \mathbb{C}$ is said to be k-fold symmetric in $\mathbb{D}$ if for every $z \in \mathbb{D}$

$$f(e^{2\pi i/k} z) = e^{2\pi i/k} f(z).$$

Denote by $\mathcal{S}^{(k)}$ the subclass of functions $f \in \mathcal{S}$ that are k-fold symmetric in $\mathbb{D}$. The following theorem gives closure property under certain transformations.

Theorem 2.5. *Let f be in the class $\mathcal{ST}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. Then, the following holds:*

(i) The k-th root transform g defined by $g(z) = \sqrt[k]{f(z^k)}$ is in the class $\mathcal{ST}^{(k)} := \mathcal{ST} \cap \mathcal{S}^{(k)}$.

(ii) The function $g : \mathbb{D} \to \mathbb{C}$ defined by

$$g(z) := \overline{f(\overline{z})} = z + \sum_{n=2}^{\infty} \overline{a_n} z^n$$

is in the class $\mathcal{ST}$.

(iii) The function $g : \mathbb{D} \to \mathbb{C}$ defined by $g(z) := f(rz)/r \quad (0 < r < 1)$ is in the class $\mathcal{ST}$.

2.1 Coefficients Estimates

As an application of Theorem 2.4, we obtain the bounds for the Taylor coefficients of starlike functions. The idea is to relate the coefficients of the given function $f \in \mathcal{ST}$ to the coefficients c_k of functions $p \in \mathcal{P}$ of functions with positive real part and use the known estimate $|c_k| \leqslant k$ where $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$.

Theorem 2.6. *If the function f is in the class $\mathcal{ST}$ and has the Taylor series expansion $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then $|a_k| \leqslant k$ for each integer $k \geqslant 2$ with equality if f is a rotation of the Koebe function κ.*

Proof. If $f \in \mathcal{ST}$, then, by Theorem 2.4, the function $p : \mathbb{D} \to \mathbb{C}$ defined by

$$p(z) := \frac{zf'(z)}{f(z)}$$

belongs to the class $\mathcal{P}$. By writing $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$, we see that $zf'(z) = f(z)p(z)$ becomes

$$z + \sum_{k=2}^{\infty} k a_k z^k = \left(z + \sum_{k=1}^{\infty} a_k z^k\right)\left(1 + \sum_{k=1}^{\infty} c_k z^k\right).$$

Equating coefficients of z^k in the above equation gives

$$k a_k = a_k + \sum_{l=1}^{k-1} a_{k-l} c_l \quad (k \geqslant 2)$$

where $a_1 = 1$ or

$$(k-1)a_k = \sum_{l=1}^{k-1} a_{k-l} c_l \quad (k \geqslant 2).$$

For $k = 2$, we have $a_2 = c_1$, and hence $|a_2| = |c_1| \leqslant 2$.

Assume that $|a_l| \leqslant l$ for $2 \leqslant l \leqslant k-1$. Then

$$|(k-1)a_k| \leqslant \sum_{l=1}^{k-1} |a_{k-l}|\,|c_l| \leqslant 2\sum_{l=1}^{k-1}(k-l) = (k-1)k.$$

Hence, by the principle of mathematical induction, $|a_k| \leqslant k$. Equality clearly holds for the rotations κ_θ of the Koebe function κ defined by

$$\kappa_\theta(z) = e^{-i\theta}\kappa(e^{i\theta}z) \quad (\theta \in [0, 2\pi]). \qquad \square$$

Theorem 2.7 Clunie's Theorem. *If the function f is in the class $\mathcal{ST}$ and has the Taylor series expansion $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then*

$$(n-1)^2 |a_n|^2 \leqslant 4\left(1 + \sum_{k=2}^{n-1} k\,|a_k|^2\right) \tag{2.6}$$

and $|a_n| \leqslant n$ for each integer $n \geqslant 2$ with equality if f is a rotation of the Koebe function.

Proof. Since $f \in \mathcal{ST}$, then, by Theorem 2.4, the function $p : \mathbb{D} \to \mathbb{C}$ defined by

$$p(z) := \frac{zf'(z)}{f(z)} \tag{2.7}$$

belongs to the class $\mathcal{P}$. The function $w : \mathbb{D} \to \mathbb{D}$ defined by

$$w(z) = \frac{p(z)-1}{p(z)+1} \tag{2.8}$$

belongs to the class $\mathcal{B}_0$. Using (2.7) in (2.8), we get

$$w(z) = \frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} + 1} = \frac{zf'(z) - f(z)}{zf'(z) + f(z)}$$

or

$$(zf'(z) + f(z))\,w(z) = zf'(z) - f(z).$$

By writing $w(z) = \sum_{k=1}^{\infty} w_k z^k$, the last equation becomes

$$\left(2z + \sum_{k=2}^{\infty}(k+1)a_k z^k\right) \cdot \sum_{k=1}^{\infty} w_k z^k = \sum_{k=2}^{\infty}(k-1)a_k z^k. \tag{2.9}$$

It follows by equating the coefficients of z^n on both sides of (2.9) that, for $n \geqslant 2$,

$$2w_{n-1} + 3a_2 w_{n-2} + \cdots + n a_{n-1} w_1 = (n-1)a_n. \tag{2.10}$$

This means that the coefficient a_n on the right of (2.10) depends only on $a_2, \ldots, a_{n-1}$ on the left of (2.10). Hence, for $n \geqslant 2$, we can write

$$\left(2z + \sum_{k=2}^{n-1}(k+1)a_k z^k\right) w(z) = \sum_{k=2}^{n}(k-1)a_k z^k + \sum_{k=n+1}^{\infty} b_k z^k. \tag{2.11}$$

Writing $z = re^{i\theta}$, integrating the square of the modulus of both sides of (2.11) with respect to θ, and then using the fact that $|w(z)| < 1$ in $|z| < 1$, we get, using (1.1),

$$\begin{aligned}
&\sum_{k=2}^{n}(k-1)^2 |a_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |b_k|^2 r^{2k} \\
&\quad = \frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{k=2}^{n}(k-1)a_k z^k + \sum_{k=n+1}^{\infty} b_k z^k\right|^2 d\theta \\
&\quad = \frac{1}{2\pi}\int_0^{2\pi} \left|\left(2z + \sum_{k=2}^{n-1}(k+1)a_k z^k\right) w(z)\right|^2 d\theta \\
&\quad \leqslant \frac{1}{2\pi}\int_0^{2\pi} \left|2z + \sum_{k=2}^{n-1}(k+1)a_k z^k\right|^2 d\theta \\
&\quad \leqslant 4r^2 + \sum_{k=2}^{n-1}(k+1)^2 |a_k|^2 r^{2k}.
\end{aligned}$$

Letting $r \to 1-$, we find that

$$\begin{aligned}
\sum_{k=2}^{n}(k-1)^2 |a_k|^2 &\leqslant \sum_{2}^{n}(k-1)^2 |a_k|^2 + \sum_{k=n+1}^{\infty} |b_k|^2 \\
&\leqslant 4 + \sum_{k=2}^{n-1}(k+1)^2 |a_k|^2
\end{aligned}$$

or equivalently

$$\begin{aligned}
(n-1)^2 |a_n|^2 &\leqslant 4 + \sum_{k=2}^{n-1}\left((k+1)^2 - (k-1)^2\right)|a_k|^2 \\
&= 4\left(1 + \sum_{k=2}^{n-1} k\,|a_k|^2\right).
\end{aligned}$$

Equality clearly holds for the rotations κ_θ of the Koebe function κ defined by

$$\kappa_\theta(z) = e^{-i\theta}\kappa(e^{i\theta}z) = \frac{z}{(1 - e^{i\theta}z)^2} \quad (\theta \in [0, 2\pi]).$$

The inequality $|a_n| \leqslant n$ follows from the above inequality by the principle of mathematical induction. For $n = 2$, the inequality (2.6) reduces to $|a_2|^2 \leqslant 4$ and this shows that $|a_2| \leqslant 2$. Assume that $|a_n| \leqslant n$ holds for all $n = 2, 3, \ldots, m-1$. For $n = m$, we have, using (2.6),

$$\begin{aligned}
(m-1)^2 |a_m|^2 &\leqslant 4\left(1 + \sum_{k=2}^{m-1} k\,|a_k|^2\right) \\
&\leqslant 4\left(1 + \sum_{k=2}^{m-1} k^3\right) = (m-1)^2 m^2
\end{aligned}$$

and so $|a_m| \leqslant m$. $\square$

Theorem 2.8. *If the function f in the class $\mathcal{ST}$ maps $\mathbb{D}$ onto a domain of area Δ and has the Taylor series expansion $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then*

$$|a_n| \leqslant \frac{2}{n-1}\sqrt{\frac{\Delta}{\pi}} \quad (n \geqslant 2).$$

In particular, if f is in the class $\mathcal{ST}$ and $|f(z)| \leqslant 1$, then $|a_n| \leqslant 2/(n-1)$ for $n \geqslant 2$.

The image $f(\mathbb{D})$ of a function $f \in \mathcal{S}$ contains the disk $\mathbb{D}_{1/4}$ and therefore the function f has an inverse F from a domain containing the disk $\mathbb{D}_{1/4}$. Let

$$F(w) = w + \gamma_2 w^2 + \gamma_3 w^3 + \cdots,$$

K be the inverse of the following rotation of the Koebe function

$$w = \widetilde{\kappa}(z) = -\kappa(-z) = \frac{z}{(1+z)^2} = \frac{1}{4}\left(1 - \left(\frac{1-z}{1+z}\right)^2\right).$$

Solving for z, we get

$$\begin{aligned} z &= \frac{1-\sqrt{1-4w}}{1+\sqrt{1-4w}} = \frac{1-2w-\sqrt{1-4w}}{2w} \\ &= w + 2w^2 + 5w^3 + 14w^4 + 42w^5 + 132w^6 + \cdots. \end{aligned}$$

If we write

$$K(w) = w + K_2 w^2 + K_3 w^3 + \cdots,$$

a computation shows that

$$K_n = \frac{(2n)!}{n!(n+1)!}.$$

Löwner showed that $|\gamma_n| \leqslant K_n$ for each $n \geqslant 2$. The proof of this result requires Löwner theory but the same result for starlike functions does not require his theory.

Theorem 2.9 [23]. *If a function f is in the class $\mathcal{ST}$ and its inverse F is given by*

$$F(w) = w + \gamma_2 w^2 + \gamma_3 w^3 + \cdots,$$

then

$$|\gamma_n| \leqslant K_n = \frac{(2n)!}{n!(n+1)!}$$

where K_n is the nth coefficients of the inverse of $-\kappa(-z)$ where κ is the Koebe function. The equality holds for the inverse of the Koebe function.

3. Starlike Functions of Order α

The class of starlike functions is defined geometrically by requiring that the functions map the unit disk onto domain starlike with respect to the origin. However, the class of starlike functions can be generalized in several ways by using the analytic characterization of the class. Recall that a function f is in $\mathcal{ST}$ if the quantity $zf'(z)/f(z)$ takes values in the right half-plane. One possible generalization is to the replace the right-half-plane by other half-planes.

Definition 3.1. *A function $f \in \mathcal{A}$ is starlike of order α, $0 \leqslant \alpha < 1$, if*

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha \quad (z \in \mathbb{D}).$$

The class of all starlike functions of order α is denoted by $\mathcal{ST}(\alpha)$.

If $f \in \mathcal{A}$ is starlike of order α, then it is clearly starlike; in other words, $\mathcal{ST}(\alpha) \subset \mathcal{ST}$. More generally, for $0 < \beta < \alpha$, we have $\mathcal{ST}(\alpha) \subset \mathcal{ST}(\beta)$. The following theorem gives a connection between two subclasses of starlike functions.

Theorem 3.1. *If f is in the class $\mathcal{ST}(\alpha)$, $0 \leqslant \alpha < 1$, then the function $g : \mathbb{D} \to \mathbb{C}$ defined by*

$$g(z) = z\left(\frac{f(z)}{z}\right)^{\frac{1-\beta}{1-\alpha}}, \quad 0 \leqslant \beta < 1,$$

is in the class $\mathcal{ST}(\beta)$.

Proof. The result follows from

$$\frac{1}{1-\alpha}\left(\frac{zf'(z)}{f(z)} - \alpha\right) = \frac{1}{1-\beta}\left(\frac{zg'(z)}{g(z)} - \beta\right). \qquad \square$$

If f is in the class $\mathcal{ST}(\alpha)$, then, by Theorem 3.1, the function $g : \mathbb{D} \to \mathbb{C}$ defined by

$$g(z) = z\left(\frac{f(z)}{z}\right)^{\frac{1}{1-\alpha}}$$

is in the class $\mathcal{ST}$. This relation gives the distortion bounds for f from the corresponding bounds for g. It also implies that the logarithmic coefficient of f is the product of $1-\alpha$ and the corresponding logarithmic coefficient of g.

Theorem 3.2. *Let $0 \leqslant \alpha < 1$ and γ_k's be real numbers such that $\gamma_k \geqslant 0$, and $\sum_1^n \gamma_k \leqslant 1$. If f_k is in the class $\mathcal{ST}(\alpha)$ for each k, then the function $g : \mathbb{D} \to \mathbb{C}$ defined by*

$$g(z) = z\prod_{k=1}^{n}\left(\frac{f_k(z)}{z}\right)^{\gamma_i}$$

belongs to the class $\mathcal{ST}(\alpha)$.

Proof. Since each f_k is univalent, the function g is analytic and indeed g is in the class $\mathcal{A}$. A computation shows that

$$\operatorname{Re}\frac{zg'(z)}{g(z)} = 1 + \sum_{k=1}^{n} \gamma_k \operatorname{Re}\left(\frac{zf_k'(z)}{f_k(z)} - 1\right)$$
$$= 1 + \sum_{k=1}^{n} \gamma_k(\alpha - 1) \geqslant \alpha,$$

proving $g \in \mathcal{ST}(\alpha)$. □

An analytic function p is subordinate to another analytic function q, written $p \prec q$, if there is a mapping $w : \mathbb{D} \to \mathbb{D}$ with $w(0) = 0$ such that $p = q \circ w$. Indeed, a function $p \in \mathcal{P}$ if and only if $p(z) \prec (1+z)/(1-z)$. For a function p in the class $\mathcal{P}(\alpha)$, the function q defined by

$$q(z) = \frac{p(z) - \alpha}{1 - \alpha}$$

belongs to $\mathcal{P}$ and so $q(z) \prec (1+z)/(1-z)$. This gives

$$p(z) \prec \alpha + (1-\alpha)\frac{1+z}{1-z} = \frac{1+(1-2\alpha)z}{1-z}.$$

Therefore, a function f in the class $\mathcal{A}$ belong to the class $\mathcal{ST}(\alpha)$ if and only if

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\alpha)z}{1-z}. \tag{3.1}$$

Example 3.1 Generalized Koebe function. For $0 \leqslant \alpha < 1$, consider the function $\kappa_\alpha : \mathbb{D} \to \mathbb{C}$ defined by

$$\kappa_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}}$$

known as the generalized Koebe function of order α. Since

$$\operatorname{Re}\frac{z\kappa_\alpha'(z)}{\kappa_\alpha(z)} = 1 + \operatorname{Re}\frac{2(1-\alpha)z}{1-z} = \alpha + (1-\alpha)\operatorname{Re}\frac{1+z}{1-z} > \alpha,$$

it follows that κ_α is in the class $\mathcal{ST}(\alpha)$. Also, since

$$\frac{z\kappa_\alpha'(z)}{\kappa_\alpha(z)} = \frac{1+(1-2\alpha)z}{1-z},$$

we see from (3.1) that f is in the class $\mathcal{ST}(\alpha)$ if and only if

$$\frac{zf'(z)}{f(z)} \prec \frac{z\kappa_\alpha'(z)}{\kappa_\alpha(z)}.$$

Theorem 3.3. *Let f be in the class $\mathcal{A}$, $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $0 \leqslant \alpha < 1$. If*

$$\sum_{k=2}^{\infty} (k-\alpha)\,|a_k| \leqq 1 - \alpha, \tag{3.2}$$

then f is in the class $\mathcal{ST}(\alpha)$. If $a_k \leqslant 0$ for all k, then the above condition (3.2) *is a necessary condition for f to belong to the class $\mathcal{ST}(\alpha)$.*

Proof. Since $k \leqslant (k-\alpha)/(1-\alpha)$ for each $k \geqslant 1$, by (3.2), we have

$$\sum_{k=2}^{\infty} k\,|a_k| \leqslant \sum_{k=2}^{\infty} \frac{k-\alpha}{1-\alpha}\,|a_k| \leqslant 1.$$

The function f is in the class $\mathcal{S}$ and so $f(z) \neq 0$ for $z \neq 0$. We also have

$$|zf'(z) - f(z)| - (1-\alpha)|f(z)|$$
$$= \left|\sum_{k=2}^{\infty}(k-1)a_k z^k\right| - (1-\alpha)\left|z + \sum_{k=2}^{\infty} a_k z^k\right|$$
$$\leqslant \sum_{k=2}^{\infty}(k-1)|a_k||z|^k - (1-\alpha)\left(|z| - \sum_{k=2}^{\infty}|a_k||z|^k\right)$$
$$\leqslant |z|\left(\sum_{k=2}^{\infty}(k-\alpha)|a_k| - (1-\alpha)\right) \leqslant 0.$$

Therefore, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leqslant 1 - \alpha$$

and hence

$$-\operatorname{Re}\left(\frac{zf'(z)}{f(z)} - 1\right) \leqslant \left|\frac{zf'(z)}{f(z)} - 1\right| \leqslant 1 - \alpha.$$

This proves that the function f is in the class $\mathcal{ST}(\alpha)$.

Suppose that the function f is in the class $\mathcal{ST}(\alpha)$ and $a_k \leqslant 0$ for all k. Then, for $z = r$ with $0 < r < 1$, we have

$$\alpha < \operatorname{Re}\frac{zf'(z)}{f(z)} = \frac{rf'(r)}{f(r)}. \tag{3.3}$$

Since $f \in \mathcal{S}$, we have $f'(z) \neq 0$ and so $f'(r)$ is positive for all r or negative for all r. Since $f'(0) = 1$, it follows that $f'(r) > 0$. This together with (3.3) shows that $f(r) > 0$. Therefore, we can write (3.3) as

$$0 < f'(r) - \alpha\frac{f(r)}{r} = 1 - \alpha + \sum_{k=2}^{\infty}(k-\alpha)a_k r^{k-1}$$
$$= 1 - \alpha - \sum_{k=2}^{\infty}(k-\alpha)|a_k| r^{k-1}.$$

Letting $r \to 1-$, we get $\sum_{k=2}^{\infty}(k-\alpha)|a_k| \leqslant 1-\alpha$. □

Example 3.2. Let $f_n : \mathbb{D} \to \mathbb{C}$ be defined by

$$f_n(z) = z - \frac{n-\alpha}{1-\alpha} z^n.$$

By Theorem 3.3, this function f_n is in the class $\mathcal{ST}(\alpha)$. It can be shown that the function $f : \mathbb{D} \to \mathbb{C}$ defined by $f(z) = z + A_n z^n$ is starlike of order α if and only if

$$|A_n| \leqslant \frac{n-\alpha}{1-\alpha}.$$

Several results for starlike functions of order α can be found in [33].

3.1 Representation Theorem

For a function p is in the class $\mathcal{P}$, the Herglotz Representation Theorem shows that there is a non-decreasing function μ on $[0, 2\pi]$ with $\int_0^{2\pi} d\mu(t) = 1$ satisfying

$$p(z) = \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t). \qquad (3.4)$$

We use this to give a representation theorem for starlike functions of order α.

Theorem 3.4. *A function $f \in \mathcal{A}$ is in the class $\mathcal{ST}(\alpha)$, $0 \leqslant \alpha < 1$, if and only if there is a function p in the class $\mathcal{P}$ such that*

$$f(z) = z \exp\left((1-\alpha)\int_0^z \frac{p(w)-1}{w} dw\right). \qquad (3.5)$$

Proof. Suppose that the function f is in the class $\mathcal{ST}(\alpha)$. Then, the function $p : \mathbb{D} \to \mathbb{C}$ defined by

$$p(z) = \frac{1}{1-\alpha}\left(\frac{zf'(z)}{f(z)} - \alpha\right) \quad (|z| < 1)$$

belongs to the class $\mathcal{P}$. Since the function f is in the class $\mathcal{S}$, it follows that $f(z)/z$ is analytic and non-vanishing in $\mathbb{D}$ and so $\log(f(z)/z)$ is analytic in $\mathbb{D}$. Integrating from 0 to z, the equation

$$(1-\alpha)\frac{p(w)-1}{w} = \frac{f'(w)}{f(w)} - \frac{1}{w}$$

gives

$$(1-\alpha)\int_0^z \frac{p(w)-1}{w} dw = \log\left(\frac{f(w)}{w}\right)\Bigg|_0^z = \log\left(\frac{f(z)}{z}\right).$$

From this, (3.5) follows. Notice that both $f(z)/z$ and $(p(z)-1)/z$ have removable singularities at the origin.

If (3.5) holds for some $p \in \mathcal{P}$, then $f(z) \neq 0$ for $z \neq 0$ and so $\log(f(z)/z)$ is analytic and

$$\log\left(\frac{f(z)}{z}\right) = (1-\alpha)\int_0^z \frac{p(w)-1}{w} dw.$$

By differentiating this and rewriting, we see that

$$\frac{zf'(z)}{f(z)} = \alpha + (1-\alpha)p(z).$$

Since the function p is in the class $\mathcal{P}$, the above equation shows that

$$\operatorname{Re}\frac{zf'(z)}{f(z)} = \alpha + (1-\alpha)\operatorname{Re} p(z) > \alpha,$$

that is, the function f is in the class $\mathcal{ST}(\alpha)$. □

Theorem 3.5 Representation theorem. *Let $f \in \mathcal{A}$. The function f is in the class $\mathcal{ST}(\alpha)$, $0 \leqslant \alpha < 1$, if and only if there is a non-decreasing function μ on $[0, 2\pi]$ with $\int_0^{2\pi} d\mu(t) = 1$ such that*

$$f(z) = z \exp\left(-2(1-\alpha)\int_0^{2\pi} \log(1 - e^{-it}z) d\mu(t)\right). \qquad (3.6)$$

Proof. By the Herglotz representation theorem, for a function p belonging to the class $\mathcal{P}$, there is a non-decreasing function μ on $[0, 2\pi]$ with $\int_0^{2\pi} d\mu(t) = 1$ satisfying

$$p(z) = \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t). \qquad (3.7)$$

Then we have

$$\begin{aligned}\int_0^z \frac{p(w)-1}{w} dw &= 2\int_0^z \int_0^{2\pi} \frac{e^{-it}}{1 - e^{-it}w} d\mu(t) dw \\ &= 2\int_0^{2\pi} \int_0^z \frac{e^{-it}}{1 - e^{-it}w} dw d\mu(t) \\ &= -2\int_0^{2\pi} \log(1 - e^{-it}z) d\mu(t).\end{aligned}$$

By using (3.5), we have

$$f(z) = z \exp\left(-2(1-\alpha)\int_0^{2\pi} \log(1 - e^{-it}z) d\mu(t)\right).$$

The converse follows by Theorem 3.4 as the function p given by (3.7) is in the class $\mathcal{P}$. □

Theorem 3.6 Marx–Strohhäcker Theorem. *If the function f is in the class $\mathcal{ST}(\alpha)$, $0 \leqslant \alpha < 1$, then*

$$\operatorname{Re}\left(\frac{f(z)}{z}\right)^{\frac{1}{2(1-\alpha)}} > \frac{1}{2}.$$

Equivalently,

$$\frac{zf'(z)}{f(z)} \prec \frac{z\kappa_\alpha'(z)}{\kappa_\alpha(z)} \Longrightarrow \frac{f(z)}{z} \prec \frac{\kappa_\alpha(z)}{z}$$

where κ_α is the generalized Koebe function of order α.

Proof. If $f \in \mathcal{ST}(\alpha)$, then, by Theorem 3.5, there is a non-decreasing function μ on $[0, 2\pi]$ with $\int_0^{2\pi} d\mu(t) = 1$ such that

$$f(z) = z \exp\left(-2(1-\alpha)\int_0^{2\pi} \log(1 - e^{-it}z) d\mu(t)\right).$$

This gives

$$\left(\frac{f(z)}{z}\right)^{\frac{1}{2(1-\alpha)}} = \exp\left(-\int_0^{2\pi} \log(1 - e^{-it}z) d\mu(t)\right).$$

Since $\operatorname{Re} e^z = e^{\operatorname{Re} z}\cos(\Im z) \geqslant -e^{\operatorname{Re} z}$ and $\operatorname{Re}\log z = \log|z|$, the above equations shows that

$$\operatorname{Re}\left(\frac{f(z)}{z}\right)^{\frac{1}{2(1-\alpha)}} \geqslant \exp\left(-\int_0^{2\pi}\log|1-e^{-it}z|d\mu(t)\right).$$

Since $|1-e^{-it}z| \leqslant 2$, w have $\log|1-e^{-it}z| \leqslant \log 2$ and so

$$\int_0^{2\pi}\log|1-e^{-it}z|d\mu(t) \leqslant \log 2.$$

Hence we get

$$\operatorname{Re}\left(\frac{f(z)}{z}\right)^{\frac{1}{2(1-\alpha)}} \geqslant e^{-\log 2} = \frac{1}{2}.$$

The minimum principle for harmonic functions shows the inequality is actually strict.

For the generalized Koebe function κ_α, we have

$$\operatorname{Re}\left(\frac{\kappa_\alpha(z)}{z}\right)^{\frac{1}{2(1-\alpha)}} = \operatorname{Re}\frac{1}{1-z} \to \frac{1}{2}$$

as $z \to -1$ and so the constant $1/2$ cannot be improved.

In terms of subordination, by Example 3.1, we know that f is in the class $\mathcal{ST}(\alpha)$ if and only if

$$\frac{zf'(z)}{f(z)} \prec \frac{z\kappa'_\alpha(z)}{\kappa_\alpha(z)}.$$

The inequality

$$\operatorname{Re}\left(\frac{f(z)}{z}\right)^{\frac{1}{2(1-\alpha)}} > \frac{1}{2}$$

is the same as the subordination

$$\left(\frac{f(z)}{z}\right)^{\frac{1}{2(1-\alpha)}} \prec \frac{1}{1-z}$$

or

$$\frac{f(z)}{z} \prec \frac{1}{(1-z)^{2(1-\alpha)}} = \frac{\kappa_\alpha(z)}{z}. \qquad \square$$

The next corollary gives the growth, distortion and covering estimates for starlike functions of order α.

Corollary 3.1. *If the function f is in the class $\mathcal{ST}(\alpha)$, then, for $|z| = r < 1$,*

$$\frac{r}{(1+r)^{2(1-\alpha)}} \leqslant |f(z)| \leqslant \frac{r}{(1-r)^{2(1-\alpha)}}$$

and

$$\frac{1-(1-2\alpha)r}{(1+r)^{3-2\alpha}} \leqslant |f'(z)| \leqslant \frac{1+(1-2\alpha)r}{(1-r)^{3-2\alpha}}.$$

These inequalities are sharp. Also, $f(\mathbb{D}) \supset \{w \in \mathbb{C} : |w| < 2^{-2(1-\alpha)}\}$.

Corollary 3.2 Marx–Strohhäcker Theorem. *If the function f is in the class $\mathcal{ST}$, then we have*

$$\operatorname{Re}\sqrt{\frac{f(z)}{z}} > \frac{1}{2}.$$

Equivalently,

$$\frac{zf'(z)}{f(z)} \prec \frac{z\kappa'(z)}{\kappa(z)} \Longrightarrow \frac{f(z)}{z} \prec \frac{\kappa(z)}{z}$$

where κ is the Koebe function.

The subordination obtained in Corollary 3.2 implies that

$$\log\frac{f(z)}{z} \prec \log\frac{1}{(1-z)^2} = \log\frac{\kappa(z)}{z}.$$

This can be used to prove the bound for logarithmic coefficients of f. We can also prove Theorem 2.6 using Corollary 3.2. Since $\operatorname{Re}\sqrt{f(z)/z} > 1/2$ for $f \in \mathcal{ST}$, we see that $f(z) = zp(z)^2$ where $p \in \mathcal{P}(1/2)$. By comparing the coefficients of both sides and using the bounds $|c_k| \leqslant 1$ for $p \in \mathcal{P}(1/2)$, we get $|a_k| \leqslant k$.

As an application of the Marx–Strohhäcker Theorem (Theorem 3.6), we prove the following corollary.

Theorem 3.7. *If the function f is in the class $\mathcal{ST}(\alpha)$, $0 \leqslant \alpha < 1$, then*

$$\left|\arg\left(\frac{f(z)}{z}\right)\right| \leqslant 2(1-\alpha)\sin^{-1}(|z|).$$

Proof. For $f \in \mathcal{ST}(\alpha)$, the Marx–Strohhäcker Theorem (Theorem 3.6) shows that

$$\frac{f(z)}{z} \prec \frac{1}{(1-z)^{2(1-\alpha)}}$$

and hence

$$\left|\left(\frac{z}{f(z)}\right)^{\frac{1}{2(1-\alpha)}} - 1\right| \leqslant |z|. \qquad (3.8)$$

If $|w-1| \leqslant r$, then, $|\arg w| \leqslant \sin^{-1} r$. Using this inequality and (3.8), we have

$$\frac{1}{2(1-\alpha)}\left|\arg\left(\frac{f(z)}{z}\right)\right| = \left|\arg\left(\frac{f(z)}{z}\right)^{\frac{1}{2(1-\alpha)}}\right| \leqslant \sin^{-1}(|z|),$$

proving the result. $\square$

Corollary 3.3. *If the function f is in the class $\mathcal{ST}$, then*

$$\left|\arg\left(\frac{f(z)}{z}\right)\right| \leqslant 2\sin^{-1}(|z|).$$

The above corollary should be compared with the corresponding result for the function $f \in \mathcal{S}$.

Theorem 3.8. *If the function f is in the class $\mathcal{S}$, then the following sharp inequalities holds:*

$$\left|\arg\left(\frac{f(z)}{z}\right)\right| \leqslant \log\frac{1+|z|}{1-|z|}$$

and

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| \leqslant \log\frac{1+|z|}{1-|z|}.$$

Not every univalent function is starlike (of order α, $0 \leqslant \alpha < 1$). However, we can find some $\rho_f = \rho_f(\alpha) > 0$ such that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha$$

for all $z \in \mathbb{D}_{\rho_f}$. The largest such ρ_f is called the radius of starlikeness of order α of f. The number $R_{\mathcal{ST}(\alpha)}(\mathcal{S}) := \inf\{\rho_f : f \in \mathcal{S}\}$ is known as the radius of starlikeness of order α of the class $\mathcal{S}$. The radius of starlikeness of order 0 of the class $\mathcal{S}$ is simply called as the radius of starlikeness of the class $\mathcal{S}$. More generally, for a subclass $\mathcal{M}$ of $\mathcal{A}$, the radius of starlikeness of the class $\mathcal{M}$ is defined by $R_{\mathcal{ST}(\alpha)}(\mathcal{M}) := \inf\{\rho_f : f \in \mathcal{M}\}$. For each $f \in \mathcal{M}$, it can be seen easily that the function $f_\rho : \mathbb{D} \to \mathbb{C}$ defined by $f_\rho(z) = f(\rho z)/\rho$ is starlike of order α for each $0 < \rho \leqslant R_{\mathcal{ST}(\alpha)}(\mathcal{M})$.

Corollary 3.4. *The radius of starlikeness of the class $\mathcal{S}$ is* $\tanh(\pi/4) \approx 0.655$.

Proof. Given $f \in \mathcal{S}$, define the function $f_\rho : \mathbb{D} \to \mathbb{C}$ by $f_\rho(z) = f(\rho z)/\rho$. By using Theorem 3.8,

$$\left|\arg\left(\frac{zf_\rho'(z)}{f_\rho(z)}\right)\right| = \left|\arg\left(\frac{\rho zf'(\rho z)}{f(\rho z)}\right)\right| \leqslant \log\frac{1+|\rho z|}{1-|\rho z|} \leqslant \log\frac{1+|\rho|}{1-|\rho|}.$$

By exponentiating the inequality

$$\log\frac{1+|\rho|}{1-|\rho|} \leqslant \frac{\pi}{2}$$

and rewriting, we see that it is equivalent to

$$|\rho| \leqslant \frac{e^{\pi/2}-1}{e^{\pi/2}+1} = \tanh\left(\frac{\pi}{4}\right).$$

Therefore, we have, for all $0 < \rho \leqslant \tanh(\pi/4) \approx 0.655$,

$$\left|\arg\left(\frac{zf_\rho'(z)}{f_\rho(z)}\right)\right| < \frac{\pi}{2}$$

or

$$\operatorname{Re}\left(\frac{zf_\rho'(z)}{f_\rho(z)}\right) > 0.$$

Therefore, the radius of starlikeness of the class $\mathcal{S}$ is at least $\tanh(\pi/4)$. Verification of the sharpness is left to the reader. □

3.2 Logarithmic Coefficients

Let $f \in \mathcal{A}$, $f(z) \neq 0$ for $z \neq 0$ and the function $\log(f(z)/z)$ is analytic in $\mathbb{D}$. Let

$$\log\left(\frac{f(z)}{z}\right) = 2\sum_{k=1} \gamma_k z^k. \tag{3.9}$$

The coefficients γ_k are called the logarithmic coefficients of the function f.

Theorem 3.9. *The logarithmic coefficients γ_k of the function f belonging to the class $\mathcal{ST}(\alpha)$ satisfies the inequality $|\gamma_k| \leqslant (1-\alpha)/k$ with equality for the generalized Koebe function κ_α.*

Proof. By (3.6), we have

$$\begin{aligned}\log\left(\frac{f(z)}{z}\right) &= -2(1-\alpha)\int_0^{2\pi} \log(1-e^{-it}z)\,d\mu(t)\\ &= \int_0^{2\pi}\left(\sum_{k=1}^{\infty}\frac{2(1-\alpha)e^{ikt}}{k}z^k\right)d\mu(t)\\ &= 2(1-\alpha)\sum_{k=1}^{\infty}\left(\int_0^{2\pi}\frac{e^{ikt}}{k}d\mu(t)\right)z^k.\end{aligned}$$

Therefore, the logarithmic coefficients γ_k of the function f are given by

$$\gamma_k = (1-\alpha)\int_0^{2\pi}\frac{e^{ikt}}{k}d\mu(t), \quad k \geqslant 1,$$

and hence

$$|\gamma_k| \leqslant \frac{1-\alpha}{k}\int_0^{2\pi} d\mu(t) = \frac{1-\alpha}{k}.$$

For the generalized Koebe function κ_α, we have

$$\log\left(\frac{\kappa_\alpha(z)}{z}\right) = -2(1-\alpha)\log(1-z) = 2(1-\alpha)\sum_{k=1}^{\infty}\frac{z^k}{k}$$

and so $\gamma_k = (1-\alpha)/k$. □

The bounds for logarithmic coefficients gives the following corollary.

Corollary 3.5. *The coefficients a_k of the function f in the class $\mathcal{ST}(\alpha)$ satisfies the inequality $|a_k| \leqslant A_k$, where*

$$A_k := \frac{1}{(k-1)!}\prod_{l=2}^{k}(l-2\alpha).$$

The equality holds for the generalized Koebe function κ_α.

The following theorem extends Theorem 2.7 for starlike functions of order α.

Theorem 3.10 Clunie's Theorem. *If the function f is in the class $\mathcal{ST}(\alpha)$ and $f(z) = z + a_2z^2 + a_3z^3 + \cdots$, then*

$$(n-1)^2|a_n|^2 \leqslant 4(1-\alpha)\left(1-\alpha+\sum_{k=2}^{n-1}(k-\alpha)|a_k|^2\right)$$

for each integer $n \geqslant 2$ with equality if f is a rotation of the generalized Koebe function κ_α.

As in the case of starlike functions, this theorem can be used to find a sharp estimate for the coefficients a_k (by induction) of starlike functions of order α. However, we got the estimate by using logarithmic coefficients in Corollary 3.5. For some recent results on the logarithm coefficients of univalent functions as well as for starlike functions, see [37,40,41,42].

3.3 Subordination Theorems

Geometrically, it is clear that every convex function is starlike. This result is now generalized to give a sufficient condition for starlikeness for the functions in the class $\mathcal{A}_n$ of all analytic functions $f : \mathbb{D} \to \mathbb{C}$ whose Taylor's series is of the form $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots$. The idea is to apply the theory of differential subordination. Let Ω be a subset of $\mathbb{C}$. The class $\mathcal{P}_n(\Omega)$ of *admissible functions* consists of all functions $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ that satisfy the admissibility condition

$$\psi(i\rho, \sigma; z) \notin \Omega \text{ when } \rho \in \mathbb{R}, \quad \sigma \le -\frac{n}{2}(1+\rho^2), \text{and } z \in \mathbb{D}. \tag{3.10}$$

Miller and Mocanu [30] proved the following theorem.

Theorem 3.11. *Let $\psi \in \mathcal{P}_n(\Omega)$. If $p : \mathbb{D} \to \mathbb{C}$ is analytic, $p(z) = 1 + c_nz^n + \cdots$ and*

$$\psi(p(z), zp'(z); z) \in \Omega \quad (z \in \mathbb{D}),$$

then $p \in \mathcal{P}$.

As an application, we have the following theorem.

Theorem 3.12. *Let $\Omega := \mathbb{C}\backslash\{w = it : t \in \mathbb{R}, |t| \geqslant \sqrt{n(n+2)}\}$. If the function $f \in \mathcal{A}_n$ satisfies*

$$1 + \frac{zf''(z)}{f'(z)} \in \Omega, \tag{3.11}$$

then the function f is in the class $\mathcal{ST}$.

Proof. Define the function $p : \mathbb{D} \to \mathbb{C}$ by

$$p(z) := \frac{zf'(z)}{f(z)}. \tag{3.12}$$

Since $f \in \mathcal{A}_n$, and (3.11) holds, $f'(z) \neq 0$ and so $f(z) \neq 0$ for $z \neq 0$. Thus, the function p is analytic in $\mathbb{D}$ and $p(0) = 1$. From (3.12), we obtain

$$1 + \frac{zf''(z)}{f'(z)} = p(z) + \frac{zp'(z)}{p(z)} = \psi(p(z), zp'(z)) \in \Omega, \tag{3.13}$$

where

$$\psi(r, s) = r + \frac{s}{r}.$$

Whenever $\rho \in \mathbb{R}$ and $\sigma \leqslant -(1+\rho^2)/2$, the function ψ satisfies

$$\psi(i\rho, \sigma) = i\rho + \frac{\sigma}{i\rho} = i\left(\rho - \frac{\sigma}{\rho}\right)$$

and so

$$|\psi(i\rho, \sigma)| = (\rho^2 - \sigma)/|\rho| \geqslant \frac{1}{|\rho|}\left(\frac{n}{2} + \left(1 + \frac{n}{2}\right)\rho^2\right) =: \varphi(|\rho|) \geqslant \sqrt{n(n+2)}.$$

The last inequality holds because the function φ attains minimum at

$$|\rho| = \sqrt{n/(2+n)}.$$

Thus $\psi \in \mathcal{P}_1(\Omega)$. Since $\psi(p(z), zp'(z)) \in \Omega$, Theorem 3.11 shows that $p \in \mathcal{P}$ and so $f \in \mathcal{ST}$. □

Theorem 3.12 for $n = 1$ reduces to the following.

Corollary 3.6. *Let $\Omega := \mathbb{C}\backslash\{w = it : t \in \mathbb{R}, |t| \geqslant \sqrt{3}\}$. If $f \in \mathcal{A}$ satisfies*

$$1 + \frac{zf''(z)}{f'(z)} \in \Omega,$$

then the function f is in the class $\mathcal{ST}$. In particular, if either

$$\left|1 + \frac{zf''(z)}{f'(z)}\right| < \sqrt{3} \quad \text{or} \quad \left|\frac{zf''(z)}{f'(z)}\right| < 2,$$

then the function f is in the class $\mathcal{ST}$.

In fact, Miller and Mocanu [29] proved that f is starlike if $|f''(z)/f'(z)| \leqslant r_1 \approx 2.83$ where $r_1 = \sqrt{1+x_0^2}$, and x_0 is the smallest positive root of $x \sin x + \cos x = 1/e$; the sharp bound (approximately equal to 2.84116) was found by Anisiu et al. [3] as a root of certain equation.

The next theorem is a result proved by Goluzin [14, Theorem 5, p. 327] using a very involved series of inequalities. We will prove it very simply by checking an admissibility condition. This result has many applications in proving distortion properties and coefficient inequalities. The next theorem for $n = 1$ was proved earlier in Theorem 3.6 using Herglotz representation theorem.

Theorem 3.13. *Let $f \in \mathcal{A}_n$. If the function f is in the class $\mathcal{ST}(\alpha)$, $0 \leqslant \alpha < 1$, then*

$$\operatorname{Re}\left(\frac{f(z)}{z}\right)^{\frac{n}{2(1-\alpha)}} > \frac{1}{2}$$

or equivalently

$$\frac{f(z)}{z} \prec \frac{1}{(1-z^n)^{\frac{2(1-\alpha)}{n}}}.$$

The inequality is sharp for the function $f \in \mathcal{A}_n$ defined by $f(z) = \kappa_\alpha(z^n)^{1/n}$ where κ_α is the generalized Koebe function.

Proof. Define the function $p : \mathbb{D} \to \mathbb{C}$ by

$$p(z) := 2\left(\frac{f(z)}{z}\right)^{\frac{n}{2(1-\alpha)}} - 1. \qquad (3.14)$$

Since $f \in \mathcal{A}_n$, we have $p \in \mathcal{H}[1, n]$. From (3.14), we obtain

$$f(z) = z\left(\frac{p(z)+1}{2}\right)^{\frac{2(1-\alpha)}{n}}.$$

A computation gives

$$\frac{1}{1-\alpha}\left(\frac{zf'(z)}{f(z)} - \alpha\right) = 1 + \frac{2}{n}\frac{zp'(z)}{p(z)+1}$$

$$= \psi(p(z), zp'(z)) \in \Omega, \qquad (3.15)$$

where

$$\psi(r, s) = 1 + \frac{2}{n}\frac{s}{r+1} \quad \text{and} \quad \Omega = \{w \in \mathbb{C} : \operatorname{Re} w > 0\}.$$

The function ψ satisfies

$$\operatorname{Re}\psi(i\rho, \sigma) = \operatorname{Re}\left(1 + \frac{2}{n}\frac{\sigma}{1+\rho^2}(1 - i\rho)\right)$$

$$= 1 + \frac{2}{n}\frac{\sigma}{1+\rho^2} \leqslant 1 - 1 = 0,$$

or $\psi(i\rho, \sigma) \notin \Omega$ whenever $\rho \in \mathbb{R}$ and $\sigma \leqslant -n(1+\rho^2)/2$. This shows that the function ψ is in the class $\mathcal{P}_n(\Omega)$. Since $\psi(p(z), zp'(z)) \in \Omega$, Theorem 3.11 shows that the function p is in the class $\mathcal{P}$. This is equivalent to

$$\operatorname{Re}\left(\frac{f(z)}{z}\right)^{\frac{n}{2(1-\alpha)}} > \frac{1}{2}.$$

The function $f \in \mathcal{A}_n$ given by

$$f(z) = \kappa_\alpha(z^n)^{1/n} = \frac{z}{(1-z^n)^{\frac{2(1-\alpha)}{n}}}$$

is the nth root transform of the generalized Koebe function κ_α and so, by Theorem 2.5, it is starlike of order α. Indeed, it follows directly as we have

$$\operatorname{Re}\frac{zf'(z)}{f(z)} = \operatorname{Re}\frac{1 + (1-2\alpha)z^n}{1 - z^n} > \alpha.$$

Since

$$\operatorname{Re}\left(\frac{f(z)}{z}\right)^{\frac{n}{2(1-\alpha)}} = \operatorname{Re}\frac{1}{1-z^n} > \frac{1}{2},$$

and the image of $1/(1-z^n)$ is the half-plane $\{w \in \mathbb{C} : \operatorname{Re} w > 1/2\}$, the implication is sharp. □

In the proof of Theorem 3.13, we have in fact proved the following lemma.

Lemma 3.1. *If the function $p \in \mathcal{H}[1, n]$ satisfies*

$$\operatorname{Re}\left(\frac{zp'(z)}{p(z)+1}\right) > -\frac{n}{2},$$

then p is in the class $\mathcal{P}$.

For $n = 1$, Theorem 3.13 reduces to the following result.

Corollary 3.7. *If the function f is in the class $\mathcal{ST}(\alpha)$, then*

$$\operatorname{Re}\left(\frac{f(z)}{z}\right)^{\frac{1}{2(1-\alpha)}} > \frac{1}{2}. \qquad (3.16)$$

The inequality is sharp for the generalized Koebe function k_α.

It should be noted that the inequality (3.16) is equivalent to the subordination

$$\left(\frac{f(z)}{z}\right)^{\frac{1}{2(1-\alpha)}} \prec \frac{1}{1-z}$$

or to the subordination

$$\left(\frac{z}{f(z)}\right)^{\frac{1}{2(1-\alpha)}} \prec 1 - z.$$

Thus, the inequality (3.16) is equivalent to the inequality

$$\left|\left(\frac{z}{f(z)}\right)^{\frac{1}{2(1-\alpha)}} - 1\right| < 1.$$

Problem. For a complex number c with $\operatorname{Re} c > -1$ and a function $f \in \mathcal{A}$, define $F_c : \mathbb{D} \to \mathbb{C}$ by

$$F_c(z) = \frac{c+1}{z^c} \int_0^z \xi^{c-1} f(\xi) d\xi$$
$$= (c+1) \int_0^1 t^{c-1} f(tz) dt. \tag{3.17}$$

The operator $f \to F_c$ is known as the Bernardi (integral) operator and the function F_c has the series expansion given by

$$F_c(z) = z + \sum_{n=2}^{\infty} \left(\frac{c+1}{c+n}\right) a_n z^n.$$

For $c = 0$, the operator $f \to F_0$ is known as the Alexander operator and the function F_0 is given by

$$F_0(z) = \int_0^z \frac{f(\xi)}{\xi} d\xi.$$

For $c = 1$, the operator $f \to F_1$ is known as the Libera operator and the function F_1 is given by

$$F_1(z) = \frac{2}{z} \int_0^z f(\xi) d\xi.$$

In 1965, Libera [22] proved that the function F_1 is in the class $\mathcal{ST}$ if the function f is in the class $\mathcal{ST}$. For $c \in \mathbb{N}$, Bernardi [5] proved that the function F_c is in the class $\mathcal{ST}$ if the function f is in the class $\mathcal{ST}$. The converse problem was dealt in [6].

Theorem 3.14. *For $0 \leqslant \alpha < 1$ and $c \geqslant 0$, let*

$$\beta(\alpha, c) = \begin{cases} \alpha - \frac{1}{2(\alpha+c)} & (0 < \alpha + c \leqslant 1 - \alpha), \\ \alpha - \frac{\alpha+c}{2(1-\alpha)^2} & (1 - \alpha \leqslant \alpha + c). \end{cases}$$

If the function f is in the class $\mathcal{ST}(\beta(\alpha, c))$, then the function F_c is in the class $\mathcal{ST}(\alpha)$. In particular, the class $\mathcal{ST}(\alpha)$ is closed under the Bernardi integral operator $f \to F_c$.

Proof. Define the function $p : \mathbb{D} \to \mathbb{C}$ by

$$p(z) = \frac{1}{1-\alpha}\left(\frac{zF_c'(z)}{F_c(z)} - \alpha\right).$$

Since

$$(c+1)f(z) = zF_c'(z) + cF_c(z) = F_c(z)\big((1-\alpha)p(z) + \alpha + c\big),$$

we have

$$\frac{zf'(z)}{f(z)} = (1-\alpha)p(z) + \alpha + \frac{zp'(z)}{(1-\alpha)p(z) + \alpha + c}$$
$$= \psi\big(p(z), zp'(z)\big) \tag{3.18}$$

where

$$\psi(r, s) = (1-\alpha)r + \alpha + \frac{s}{(1-\alpha)r + \alpha + c}.$$

Since $f \in \mathcal{ST}(\beta(\alpha, c))$, we have $\operatorname{Re}\psi(p(z), zp'(z)) > \beta(\alpha, c)$ and so the proof is complete if we show that $\operatorname{Re}\psi(i\rho, \sigma) \leqslant \beta(\alpha, c)$ when $\rho \in \mathbb{R}$ and $\sigma \leqslant -(1+\rho^2)/2$. Clearly, for $\rho \in \mathbb{R}$ and $\sigma \leqslant -(1+\rho^2)/2$, we have

$$\operatorname{Re}\psi(i\rho, \sigma) = \alpha + \frac{(\alpha+c)\sigma}{(1-\alpha)^2\rho^2 r + (\alpha+c)^2}$$
$$\leqslant \alpha - \frac{\alpha+c}{2} \frac{(1+\rho^2)}{(1-\alpha)^2\rho^2 + (\alpha+c)^2}. \tag{3.19}$$

The function $\varphi : [0, \infty) \to \mathbb{R}$ defined by

$$\varphi(\rho) := \frac{1+\rho^2}{(1-\alpha)^2\rho^2 + (\alpha+c)^2}$$

has derivative given by

$$\varphi'(\rho) = \frac{2(c+1)(2\alpha+c-1)\rho}{\big((1-\alpha)^2\rho^2 + (\alpha+c)^2\big)^2}.$$

If $1 - \alpha \leqslant \alpha + c$, it follows that the function φ is an increasing function and so $\varphi(\rho) \leqslant 1/(1-\alpha)^2$. Similarly, if $\alpha + c \leqslant 1 - \alpha$, it follows that the function φ is a decreasing function and so $\varphi(\rho) \leqslant 1/(\alpha+c)^2$. Using these in (3.19), we see that $\operatorname{Re}\psi(i\rho, \sigma) \leqslant \beta(\alpha, c)$ when $\rho \in \mathbb{R}$ and $\sigma \leqslant -(1+\rho^2)/2$.

Since $\beta(\alpha, c) \leqslant \alpha$, the second result follows from the first. □

Solving the equation (3.17) for f in terms of F_c, we have

$$f(z) = \frac{1}{c+1}\big(zF_c'(z) + cF_c(z)\big). \tag{3.20}$$

The converse problem of studying properties of f from the properties of F_c is known as Livingston problem. Suppose that the function F_c is in the class $\mathcal{ST}(\alpha)$. Then, the corresponding function f need not be in $\mathcal{ST}(\alpha)$. However, we can find some $\rho > 0$ such that

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha$$

for all $z \in \mathbb{D}_\rho$. The largest such ρ is called the radius of starlikeness of order α of f. Indeed, Livingston [24] have shown that if F_1 is in the class $\mathcal{ST}$, then the radius of starlikeness of the function $f(z) = [zF_1(z)]'/2$ is $1/2$. Bernardi extended it for F_c when $c \in \mathbb{N}$. Our next theorem determines the radius of starlikeness of order α of f when F_c is starlike of order α and it is indeed a converse of Theorem 3.14.

Theorem 3.15. *Let $0 \leqslant \alpha < 1$ and $c > -\alpha$. If the function F_c is in the class $\mathcal{ST}(\alpha)$, the radius of starlikeness of order α of f given by* (3.20) *is*

$$R_{\mathcal{ST}(\alpha)} = \frac{1+c}{2-\alpha+\sqrt{3-2\alpha+(\alpha+c)^2}}. \tag{3.21}$$

Proof. Define the function p by

$$p(z) = \frac{zF_c'(z)}{F_c(z)}.$$

Since $F_c \in \mathcal{ST}(\alpha)$, it follows that $p \in \mathcal{P}(\alpha)$. Since

$$(c+1)f(z) = zF_c'(z) + cF_c(z) = F_c(z)(p(z)+c),$$

we have

$$\frac{zf'(z)}{f(z)} = p(z) + \frac{zp'(z)}{p(z)+c}. \tag{3.22}$$

This equation is the same as the equation (3.18) with $\alpha = 0$.

For the function p belonging to the class $\in \mathcal{P}(\alpha)$, we have, with $|z| = r < 1$,

$$\operatorname{Re} p(z) \geqslant \frac{1-(1-2\alpha)r}{1+r} \quad \text{and} \quad |p'(z)| \leqslant \frac{2(\operatorname{Re} p(z)-\alpha)}{1-r^2}.$$

By using the first of these inequalities, we get

$$|p(z)+c| \geqslant \operatorname{Re} p(z) + c \geqslant \frac{1+c+(c+2\alpha-1)r}{1+r}.$$

By (3.22), we have

$$\begin{aligned}\operatorname{Re}\frac{zf'(z)}{f(z)} - \alpha &\geqslant \operatorname{Re}(p(z)-\alpha) - \frac{|zp'(z)|}{|p(z)+c|} \\ &\geqslant \frac{1+c-2(2-\alpha)r-(1-c-2\alpha)r^2}{(1-r)\big(1+c+(c+2\alpha-1)r\big)}\operatorname{Re}(p(z)-\alpha).\end{aligned}$$

Since $r = R_{\mathcal{ST}(\alpha)}$ given by (3.21) is the root of the equation

$$1+c-2(2-\alpha)r+(1-c-2\alpha)r^2 = 0, \tag{3.23}$$

it follows that $1+c-2(2-\alpha)r+(1-c-2\alpha)r^2 \geqslant 0$ for $0 \leqslant r \leqslant R_{\mathcal{ST}(\alpha)}$. Also, for $c > -\alpha$, we have $1+c+(c+2\alpha-1)r > 0$ and hence $\operatorname{Re}(zf'(z)/f(z)) \geqslant \alpha$ for $0 \leqslant r \leqslant R_{\mathcal{ST}(\alpha)}$. Thus, the radius of starlikeness of order α of f is at least $R_{\mathcal{ST}(\alpha)}$.

The sharpness can be demonstrated using the function $F_c \in \mathcal{ST}(\alpha)$ given by

$$F_c(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \quad (0 \leqslant \alpha < 1). \qquad \square$$

For $\alpha = 0$ in Theorem 3.15, we get the following corollary.

Corollary 3.8. *Let $c > 0$. If the function F_c is in the class $\mathcal{ST}$, the radius of starlikeness of f given by* (3.20) *is*

$$R_{\mathcal{ST}(\alpha)} = \frac{1+c}{2+\sqrt{3+c^2}}.$$

Theorem 3.15 on Livingston problem can be found in Bajpai and Srivastava [4] and the proof is from Chen [9].

4. Convex Functions

Recall that a region G is **convex** if the line segment $[z_1, z_2] := \{tz_1 + (1-t)z_2 : 0 \leqslant t \leqslant 1\}$ joining any two points $z_1, z_2 \in G$ lies entirely in G. A function $f \in \mathcal{S}$ is convex if $f(\mathbb{D})$ is convex and the class of all convex functions $f \in \mathcal{S}$ is denoted by $\mathcal{CV}$.

Example 4.1. The function $\ell : \mathbb{D} \to \mathbb{C}$ defined by

$$\ell(z) = \frac{z}{1-z}$$

is in the class $\mathcal{S}$ and maps $\mathbb{D}$ onto the half-plane $\{w : \operatorname{Re} w > -1/2\}$. Since half-planes are convex, the function ℓ belongs to the class $\mathcal{CV}$ of convex functions.

Theorem 4.1 Hereditary property of convex functions. *Let $f \in \mathcal{A}$. The function $f \in \mathcal{CV}$ if and only if $f(\mathbb{D}_r)$ is convex for each r with $0 < r < 1$. Or equivalently, the function $f \in \mathcal{CV}$ if and only if the function $f_r : \mathbb{D} \to \mathbb{C}$ defined by $f_r(z) = r^{-1}f(rz)$ belongs to the class $\mathcal{CV}$ for each r with $0 < r < 1$.*

To study properties of convex functions, we first give an analytic characterization of convex functions by using the functions with positive real part.

Theorem 4.2. *Let $f \in \mathcal{A}$. The function $f \in \mathcal{CV}$ if and only if*

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \quad (z \in \mathbb{D}). \tag{4.1}$$

Equivalently, the function $f \in \mathcal{CV}$ if and only if the function $p : \mathbb{D} \to \mathbb{D}$ defined by

$$p(z) = 1 + \frac{zf''(z)}{f'(z)} \tag{4.2}$$

belongs to the class $\mathcal{P}$ of Carathéodory functions.

Let $f \in \mathcal{CV}$. By the hereditary property of convex functions (Theorem 4.1 above), the function rf_r is convex for each r with $0 < r < 1$. Thus, the function f maps $\mathbb{D}_r$ to a convex domain. Therefore, the image curve C_r of the circle $|z| = r$ bounds a convex domain. Geometrically, this means that the argument of the tangent vector to C_r is a non-decreasing function as the curve is traversed in the counterclockwise direction. In other words, $\arg(\frac{\partial}{\partial\theta}f(re^{i\theta}))$ increases as θ varies from 0 to 2π. Thus, we have

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) = \frac{\partial}{\partial\theta}\arg\left(\frac{\partial}{\partial\theta}f(re^{i\theta})\right) \geqslant 0.$$

The above theorem readily shows that the class $\mathcal{CV}$ of convex functions is a compact normal family.

If $F(z) = zf'(z)$, then we have

$$\frac{zF'(z)}{F(z)} = 1 + \frac{zf''(z)}{f'(z)}.$$

From this equation, we have the following elementary relationship between convex and starlike functions proved by Alexander [1]; a detailed analysis of this paper can be found in [18].

Theorem 4.3 Alexander's theorem. *Suppose $f \in \mathcal{A}$ is locally univalent and the function F is defined by $F(z) \equiv zf'(z)$. Then, the function $f \in \mathcal{CV}$ if and only if the function $F \in \mathcal{ST}$.*

Though the bound $|a_k| \leqslant k$ holds for convex functions, Alexander theorem easily yields the sharp bound $|a_k| \leqslant 1$.

Theorem 4.4. *If $f(z)$, given by $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, is in $\mathcal{CV}$, then for each positive integer n*

$$|a_n| \leqslant 1.$$

Further this inequality is sharp for each index n, and if equality occurs for just one $n \geqslant 2$, then $f(z)$ is a rotation of the function ℓ given by $\ell(z) = z/(1-z)$.

Since every convex function is starlike, the same distortion, growth estimates holds for convex functions. However, using Alexander theorem, we prove improved bounds for a convex function in the following theorem.

Theorem 4.5. *Let $f(z)$ be in $\mathcal{CV}$. Then*

$$\frac{r}{1+r} \leqslant |f(z)| \leqslant \frac{r}{1-r},$$

$$\frac{1}{(1+r)^2} \leqslant |f'(z)| \leqslant \frac{1}{(1-r)^2},$$

and for each $k \geqslant 2$,

$$|f^k(z)| \leqslant \frac{k!}{(1-r)^{k+1}}.$$

All of these inequalities are sharp, with equality if and only if f is some rotation of the function ℓ where $\ell(z) = z/(1-z)$.

A function $f \in \mathcal{A}$ is convex of order α, $0 \leqslant \alpha < 1$, if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{D}).$$

The class of all convex functions of order α is denoted by $\mathcal{CV}(\alpha)$. The function ℓ_α is defined by $z\ell'_\alpha(z) = k_\alpha(z)$ where k_α is the generalized Koebe function. Indeed, the function is given by

$$\ell_\alpha(z) = \begin{cases} \frac{1-(1-z)^{2\alpha-1}}{2\alpha-1} & (\alpha \neq 1/2) \\ -\ln(1-z) & (\alpha = 1/2). \end{cases}$$

Theorem 4.6. *If $f \in \mathcal{CV}(\alpha)$, then the following sharp inequalities hold:*

$$\frac{1}{(1+r)^{2(1-\alpha)}} \leqslant |f'(z)| \leqslant \frac{1}{(1-r)^{2(1-\alpha)}},$$

$$\frac{(1+r)^{2\alpha-1}-1}{2\alpha-1} \leqslant |f(z)| \leqslant \frac{1-(1-r)^{2\alpha-1}}{2\alpha-1} \quad (\alpha \neq 1/2)$$

and

$$\log(1+r) < |f(z)| < -\log(1-r) \quad (\alpha = 1/2).$$

Also, the inclusion $f(\mathbb{D}) \supset \{w \in \mathbb{C} : |w| < R\}$ holds where

$$R = \begin{cases} \frac{2^{2\alpha-1}-1}{2\alpha-1} & (\alpha \neq 1/2) \\ \log 2 & (\alpha = 1/2). \end{cases}$$

Corollary 3.5 immediately gives by making use of Alexander theorem the following result.

Corollary 4.1. *The coefficients a_k of the function $f \in \mathcal{CV}(\alpha)$ satisfies the inequality $|a_k| \leqslant A_k$ where*

$$A_k = \frac{1}{k!}\prod_{l=2}^{k}(l - 2\alpha).$$

The equality holds for the function ℓ_α.

Not every univalent function is convex (of order α). For example, Koebe function is univalent and starlike but not convex. However, we can find some $\rho_f = \rho_f(\alpha) > 0$ such that

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha$$

for all $z \in \mathbb{D}_{\rho_f}$. The largest such ρ_f is called the radius of convexity of order α of f. The number $R_{\mathcal{CV}(\alpha)}(\mathcal{S}) := \inf\{\rho_f : f \in \mathcal{S}\}$ is known as the radius of convexity of order α of the class $\mathcal{S}$. The radius of convexity of order 0 is simply called as the radius of convexity. Similarly, the number $R_{\mathcal{CV}(\alpha)}(\mathcal{ST}) := \inf\{\rho_f : f \in \mathcal{ST}\}$ is known as the radius of convexity of order α of the class $\mathcal{ST}$ of starlike functions.

Our next theorem determines the radius of convexity of order α of the class $\mathcal{S}$.

Theorem 4.7. *The radius of convexity of order α of the class $\mathcal{S}$ of univalent functions is*

$$R_{\mathcal{CV}(\alpha)}(\mathcal{S}) = \frac{1-\alpha}{2+\sqrt{3+\alpha^2}}.$$

In particular, the radius of convexity of the class $\mathcal{S}$ is $2-\sqrt{3}$.

Proof. For $f \in \mathcal{S}$, the inequality (1.1) shows that

$$1 + \operatorname{Re}\frac{zf''(z)}{f'(z)} \geqslant \frac{1+r^2}{1-r^2} - \frac{4r}{1-r^2} = \frac{1-4r+r^2}{1-r^2} =: \varphi(t).$$

Note that

$$r_0 := \frac{1-\alpha}{2+\sqrt{3+\alpha^2}}$$

is the positive root of the equation $\varphi(r_0) = \alpha$ or the equivalent quadratic equation $(1+\alpha)r^2 - 4r + (1-\alpha) = 0$. Since

$$\varphi'(r) = -4\frac{1-r+r^2}{(1-r^2)^2} < 0$$

for $0 \leqslant r < 1$, the function is decreasing and so

$$1 + \operatorname{Re}\frac{zf''(z)}{f'(z)} \geqslant \varphi(t) > \varphi(r_0) = \alpha$$

and so every function $f \in \mathcal{S}$ is convex of order α in the disc $\mathbb{D}_{r_0}$. This proves that $R_{\mathcal{CV}(\alpha)}(\mathcal{S}) \geqslant r_0$. To prove $r_0 \leqslant R_{\mathcal{CV}(\alpha)}(\mathcal{S})$, consider the Koebe function k given by

$$k(z) = \frac{z}{(1+z)^2}.$$

For this function k, we have

$$1 + \operatorname{Re}\frac{zk''(z)}{k'(z)} = \operatorname{Re}\frac{1-4z+z^2}{1-z^2} \leqslant 0$$

for $z = r$ and $r_0 < r < 1$, proving $r_0 \leqslant R_{\mathcal{CV}(\alpha)}(\mathcal{S})$. Thus the radius of convexity of order α in the class S is r_0.

We also have $R_{\mathcal{CV}(0)}(\mathcal{S}) = 1/(2+\sqrt{3}) = 2-\sqrt{3}$. □

For $f \in \mathcal{A}$, the following implications hold:

$$f \in \mathcal{CV} \quad \Rightarrow \quad \operatorname{Re}\frac{zf'(z)}{f(z)} > \frac{1}{2} \quad \Rightarrow \quad \operatorname{Re}\frac{f(z)}{z} > \frac{1}{2},$$

and

$$f \in \mathcal{CV} \quad \Rightarrow \quad \operatorname{Re}\sqrt{f'(z)} > \frac{1}{2} \quad \Rightarrow \quad \operatorname{Re}\frac{f(z)}{z} > \frac{1}{2}.$$

5. Subclasses Defined by Subordination

Various subclasses of starlike (or convex) functions are defined by restricting the range of $zf'(z)/f(z)$ (or $1 + zf''(z)/f'(z)$) to lie in a subset of the right half-plane. For example, the classes of convex functions of order α is one such class. Another prominently studied class is the class of strongly starlike functions consisting of functions $f \in \mathcal{A}$ satisfying $|\arg(zf'(z)/f(z))| < \pi\gamma/2$; these are functions f for which $zf'(z)/f(z)$ takes values in a sector in the right-half plane. Yet another class is the class of parabolic starlike functions; these are functions f for which $zf'(z)/f(z)$ takes values in the parabolic region $\Omega := \{w : |w-1| < \operatorname{Re} w\}$. Also, recall that a function f is convex if and only if zf' is starlike. We define convolution $f * g$ of two analytic functions $f, g \in \mathcal{A}$ with Taylor series $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ by $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$. In terms of convolution, we see that f is convex if and only if $f * \kappa$ is starlike (where κ is Koebe function). Instead of the studying various subclasses one by one, one can unify the classes by using convolution and subordination. Indeed, Padmanabhan and Parvatham[32] defined the class of functions $f \in \mathcal{A}$ satisfying $z(k_\alpha * f)'/(k_\alpha * f) \prec \varphi$, where $k_\alpha(z) = z/(1-z)^\alpha$, $\alpha \in \mathbb{R}$ and $\varphi \in \mathcal{CV}$. Later, Shanmugam [57] studied this class by replacing the function k_α by any fixed function $g \in \mathcal{A}$. Among other classes, he has considered the class

$$\mathcal{ST}_g(\varphi) := \left\{ f \in \mathcal{A} : \frac{z(f*g)'(z)}{(f*g)(z)} \prec \varphi(z) \right\}$$

where $g \in \mathcal{A}$ is a fixed function and φ is a convex function with positive real part. He has proved several convolution results for this class and several other related classes. The convexity of φ was essential for proving convolution theorems but not necessary for getting distortion and growth theorems.

Let φ be an analytic function with positive real part in $\mathbb{D}$, normalized by the conditions $\varphi(0) = 1$ and $\varphi'(0) > 0$, such that φ maps the unit disk $\mathbb{D}$ onto a region starlike with respect to 1 that is symmetric with respect to the real axis. Ma and Minda [25] studied the following classes:

$$\mathcal{ST}(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}$$

and

$$\mathcal{CV}(\varphi) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}.$$

These functions are called Ma-Minda starlike and convex functions respectively. For special choices of φ, these classes become well-known classes of starlike and convex functions. For $-1 \leqslant B < A \leqslant 1$, $\mathcal{ST}[A,B] := \mathcal{ST}((1+Az)/(1+Bz))$ and $\mathcal{CV}[A,B] := \mathcal{CV}((1+Az)/$

$(1 + Bz))$ are the classes of Janowski [20] starlike functions and Janowski convex functions, respectively. The particular choices of A and B in $\mathcal{ST}[A, B]$ leads to several known classes which are extensively studied in literature. For $0 \leqslant \gamma < 1$, Robertson [44] introduced the classes $\mathcal{ST}(\gamma) := \mathcal{ST}[1 - 2\gamma, -1]$ and $\mathcal{CV}(\gamma) := \mathcal{CV}[1 - 2\gamma, -1]$ of starlike and convex functions of order γ, respectively. For $\gamma = 0$, these classes reduce to the classes of starlike functions $\mathcal{ST} := \mathcal{ST}(0)$ and convex functions $\mathcal{CV} := \mathcal{CV}(0)$, respectively. The class $\mathcal{ST}_M := \mathcal{ST}[1, -(M-1)/M]$ $(M > 1/2)$ was introduced by Janowski [19]. Similarly, the classes $\mathcal{ST}[1 - \alpha, 0]$ $(0 \leqslant \alpha < 1)$ and $\mathcal{ST}[\alpha, -\alpha]$ $(0 < \alpha \leqslant 1)$ were introduced by Singh [54] and Padmanabhan [31] respectively. Brannan and Kirwan [8] and Stankiewicz [58] defined the class $\mathcal{SST}(\beta) := \mathcal{ST}(((1 + z)/(1 - z))^{\beta})$, $0 < \beta \leqslant 1$, consisting of strongly starlike functions of order β. If the function $f \in \mathcal{SST}(\beta)$, then $|\arg(zf'(z)/f(z))| < \beta\pi/2$, for $z \in \mathbb{D}$. For $k \geqslant 0$, the class $k - \mathcal{ST}$ of k-starlike functions introduced by Kanas and Wisniowska [21] consists of functions $f \in \mathcal{A}$ satisfying

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right| \quad (z \in \mathbb{D}).$$

Define $\kappa_{\varphi n}$ $(n = 2, 3, 4, \cdots)$ by $\kappa_{\varphi n}(0) = \kappa'_{\varphi n}(0) - 1 = 0$ and

$$1 + \frac{z\kappa''_{\varphi n}(z)}{\kappa'_{\varphi n}(z)} = \varphi(z^{n-1}),$$

which is clearly in the class $\mathcal{CV}(\varphi)$. We write $\kappa_{\varphi 2}$ simply as κ_{φ}, which plays the role of the Koebe function for the family $\mathcal{CV}(\varphi)$.

Similarly, we define $\ell_{\varphi n}(n = 2, 3, 4, \dots)$ by $\ell_{\varphi n}(0) = \ell'_{\varphi n}(0) - 1 = 0$ and

$$\frac{z\ell'_{\varphi n}(z)}{\ell_{\varphi n}(z)} = \varphi(z^{n-1}).$$

Then $\ell_{\varphi n}$ belongs to $\mathcal{ST}(\varphi)$ and $\ell_{\varphi 2}$, which we simply denote by ℓ_{φ}, plays the role of the Koebe function in the family $\mathcal{ST}(\varphi)$. Note that $z\kappa'_{\varphi n}(z) = \ell_{\varphi n}(z)$. This function $\ell_{\varphi,n}$ is given explicitly by

$$\ell_{\varphi n}(z) = z\exp\left(\int_0^z \frac{\varphi(w^{n-1}) - 1}{w} dw\right).$$

5.1 Growth and Distortion Theorems

The following subordination result for $\mathcal{CV}(\varphi)$ yields sharp distortion, growth, covering and rotation theorems.

Theorem 5.1. *Let $f \in \mathcal{CV}(\varphi)$. Then $zf''(z)/f'(z) \prec z\kappa''_{\varphi}(z)/\kappa'_{\varphi}(z)$ and $f'(z) \prec \kappa'_{\varphi}(z)$.*

Proof. Let $p(z) = 1 + zf''(z)/f'(z)$. Then we know that $p \prec \varphi$. Thus, $p - 1 \prec \varphi - 1$, which is the same as $zf''(z)/f'(z) \prec z\kappa''_{\varphi}(z)/\kappa'_{\varphi}(z)$. Note that $(\varphi(z) - 1)/\varphi'(0) \in \mathcal{ST}$ and $p \prec \varphi$ implies that $(p(z) - 1)/\varphi'(0) \prec (\varphi(z) - 1)/\varphi'(0)$. By using a result of Suffridge [59], we may conclude that

$$\begin{aligned}\frac{1}{\varphi'(0)}\log f'(z) &= \int_0^z \frac{p(\xi) - 1}{\xi\varphi'(0)} d\xi \\ &\prec \int_0^z \frac{\varphi(\xi) - 1}{\xi\varphi'(0)} d\xi \\ &= \frac{1}{\varphi'(0)}\log \kappa'_{\varphi}(z).\end{aligned}$$

Equivalently, $f' \prec \kappa'_{\varphi}$. This completes the proof of Theorem 5.1. □

Corollary 5.1 Distortion Theorem for $\mathcal{CV}(\varphi)$. *Assume $f \in \mathcal{CV}(\varphi)$ and $|z_0| = r < 1$. Then*

$$\kappa'_{\varphi}(-r) \leqslant |f'(z_0)| \leqslant \kappa'_{\varphi}(r).$$

Equality holds for some $z_0 \neq 0$ if and only if f is a rotation of κ_{φ}.

Corollary 5.2 Growth Theorem for $\mathcal{CV}(\varphi)$. *Let $f \in \mathcal{CV}(\varphi)$ and $|z_0| = r < 1$. Then*

$$-\kappa_{\varphi}(-r) \leqslant |f(z_0)| \leqslant \kappa_{\varphi}(r).$$

Equality holds for some $z_0 \neq 0$ if and only if f is a rotation of κ_{φ}.

Corollary 5.3 Covering Theorem for $\mathcal{CV}(\varphi)$. *Suppose $f \in \mathcal{CV}(\varphi)$. Then either f is a rotation of κ_{φ} or $f(\mathbb{D}) \supseteq \{w : |w| \leqslant -\kappa_{\varphi}(-1)\}$. Here $-\kappa_{\varphi}(-1)$ is understood to be the limit of $-\kappa_{\varphi}(-r)$ as r tends to 1.*

Note that $-\kappa_{\varphi}(-r)$ is increasing in $(0, 1)$ and bounded above by 1 (recall that each function $f \in \mathcal{CV}(\varphi)$ is normalized by $f(0) = f'(0) - 1 = 0$), so the limit of $-\kappa_{\varphi}(-r)$ exists as r tends to 1.

The following rotation theorem follows from the subordination $f' \prec \kappa'_{\varphi}$ given in Theorem 5.1.

Corollary 5.4 Rotation Theorem for $\mathcal{CV}(\varphi)$. *Let $f \in \mathcal{CV}(\varphi)$ and $|z_0| = r < 1$. Then*

$$|\arg\{f'(z_0)\}| \leqslant \max_{|z|=r} \arg\left\{\kappa'_{\varphi}(z)\right\}.$$

Equality holds for some $z_0 \neq 0$ if and only if f is a rotation of κ_φ.

Next we state corresponding results for $\mathcal{ST}(\varphi)$.

Theorem 5.2. *Let $f \in \mathcal{ST}(\varphi)$. Then $zf'(z)/f(z) \prec z\ell'_\varphi(z)/\ell_\varphi(z)$ and $f(z)/z \prec \ell_\varphi(z)/z$.*

Corollary 5.5 Growth Theorem for $\mathcal{ST}(\varphi)$. *Assume $f \in \mathcal{ST}(\varphi)$ and $|z_0| = r < 1$. Then*

$$-\ell_\varphi(-r) \leqslant |f(z_0)| \leqslant \ell_\varphi(r).$$

Equality holds for some $z_0 \neq 0$ if and only if f is a rotation of ℓ_φ.

Note that $-\ell_\varphi(-r)$ is increasing in $(0,1)$ and bounded above by 1 (since each function $f \in \mathcal{ST}(\varphi)$ is normalized by $f(0) = f'(0) - 1 = 0$), so $\lim_{r\to 1} -\ell_\varphi(-r)$ exists.

Corollary 5.6 Covering Theorem for $\mathcal{ST}(\varphi)$. *Suppose $f \in \mathcal{ST}(\varphi)$. Then either f is a rotation of ℓ_φ or $f(\mathbb{D}) \supseteq \{w : |w| \leqslant -\ell_\varphi(-1)\}$. Here $-\ell_\varphi(-1)$ is defined to be $\lim_{r\to 1} -\ell_\varphi(-r)$.*

Corollary 5.7. *Let $f \in \mathcal{ST}(\varphi)$ and $|z_0| = r < 1$. Then*

$$|\arg\{f(z_0)/z_0\}| \leqslant \max_{|z|=r} \arg\{\ell_\varphi(z)/z\}.$$

Equality holds for some $z_0 \neq 0$ if and only if f is a rotation of ℓ_φ.

In order to give a distortion theorem for the class $\mathcal{ST}(\varphi)$ we will impose two extra conditions on φ, namely, $\min_{|z|=r} |\varphi(z)| = \varphi(-r)$ and $\max_{|z|=r} |\varphi(z)| = \varphi(r)$.

Theorem 5.3 Distortion Theorem for $\mathcal{ST}(\varphi)$. *Let $\min_{|z|=r} |\varphi(z)| = \varphi(-r), \max_{|z|=r} |\varphi(z)| = \varphi(r)$, $f \in \mathcal{ST}(\varphi)$ and $|z_0| = r < 1$. Then*

$$\ell'_\varphi(-r) \leqslant |f'(z_0)| \leqslant \ell'_\varphi(r).$$

Equality holds for some $z_0 \neq 0$ if and only if f is a rotation of ℓ_φ.

Proof. Let $q(z) = zf'(z)/f(z)$. Then $f \in S^*(\varphi)$ if and only if $q \prec \varphi$. By using the subordination principle, we see that

$$\varphi(-r) = \min_{|z|=r} |\varphi(z)| \leqslant |q(z_0)| \leqslant \max_{|z|=r} |\varphi(z)| = \varphi(r).$$

Thus, from Corollary 5.5, we have for $|z_0| = r$

$$\begin{aligned}\ell'_\varphi(-r) = \varphi(-r)\ell_\varphi(-r)/(-r) &\leqslant |q(z_0) f(z_0)/z_0| \\ = |f'(z_0)| &\leqslant \varphi(r)\ell_\varphi(r)/r = \ell'_\varphi(r).\end{aligned}$$

The assertion about equality follows from Corollary 5.5. □

Open Problem 5.1. *For $n \geqslant 2$, determine the sharp bound of $|f^{(n)}(z)|$ for $f \in \mathcal{CV}(\varphi)$ and $f \in \mathcal{ST}(\varphi)$. The bounds for the cases $n = 0, 1$ are the growth and distortion theorems.*

Let $\ell_\varphi : \mathbb{D} \to \mathbb{C}$ be defined by

$$\frac{z\ell'_\varphi(z)}{\ell_\varphi(z)} = \varphi(z).$$

In Theorem 5.2 it was shown that

$$f \in \mathcal{ST}(\varphi) \Rightarrow \frac{f(z)}{z} \prec \frac{\ell_\varphi(z)}{z}.$$

In the case when φ is a convex univalent function, the following general result holds:

Theorem 5.4 Ruscheweyh [51, Theorem 1, p. 275]. *Let φ be a convex function defined in $\mathbb{D}$ with $\varphi(0) = 1$. Define F by*

$$F(z) = z\exp\left(\int_0^z \frac{\varphi(x)-1}{x}\,dx\right). \tag{5.1}$$

The function f belongs to $\mathcal{ST}(\varphi)$ if and only if for all $|s| \leqslant 1$ and $|t| \leqslant 1$,

$$\frac{sf(tz)}{tf(sz)} \prec \frac{sF(tz)}{tF(sz)}. \tag{5.2}$$

Proof. Assume that (5.1) holds and let $|s| \leqslant 1, |t| \leqslant 1, s \neq t$. The function

$$p(z) = \int_0^t \left(\frac{s}{1-sx} - \frac{t}{1-tx}\right) dx$$

is convex and univalent in $\mathbb{D}$. According to the general subordination theorem, it follows that

$$\left(\frac{zf'(z)}{f(z)} - 1\right) * p(z) \prec (G(z) - 1) * p(z).$$

For every analytic function $h(z)$ with $h(0) = 0$, we have

$$h(z) * p(z) = \int_{tz}^{sz} h(x)\frac{dx}{x}$$

and thus

$$\begin{aligned}Q(z) &\equiv \int_{sz}^{tz} \left(\frac{f'(x)}{f(x)} - \frac{1}{x}\right) dx \\ &\prec \int_{sz}^{tz} \left(\frac{G(x)-1}{x}\right) dx \equiv R(z)\end{aligned}$$

This implies that $\exp Q(z) \prec \exp R(z)$. But, by definition, $\exp R(z) = tF(sz)(sF(tz))$ and a straightforward calculation gives $\exp Q(z) = tf(sz)/(sf(tz))$. □

Estimates for the Fekete-Szegő functional for functions in $ST(\varphi)$ is given in the following theorem. This results gives sharp bounds for the first two coefficients. The coefficient problem for $f \in \mathcal{ST}(\varphi)$ is also open.

Theorem 5.5. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. *If* $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{ST}(\varphi)$, *then*

$$|a_3 - \mu a_2^2| \leqslant \frac{B_1}{2} \max\left\{1; \left|\frac{B_2}{B_1} + (1-2\mu)B_1\right|\right\}. \quad (5.3)$$

The result is sharp.

Proof. Since $f \in \mathcal{ST}(\varphi)$, there is a function w in the class Ω of Schwarz functions satisfying that

$$\frac{zf'(z)}{f(z)} = \varphi(w(z)). \quad (5.4)$$

Corresponding to the function w, define the function $p_1 : \mathbb{D} \to \mathbb{C}$ by

$$p_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \cdots \quad (5.5)$$

so that

$$w(z) = \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2}c_1 z + \frac{1}{2}\left(c_2 - \frac{1}{2}c_1^2\right) z^2 + \cdots. \quad (5.6)$$

Clearly, the function p_1 is analytic in $\mathbb{D}$ with $p_1(0) = 1$. Since $w \in \Omega$, it follows that $p_1 \in \mathcal{P}$. Using (5.6) and the Taylor series of φ given by $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$, we get

$$\varphi(w(z)) = 1 + \frac{1}{2}B_1 c_1 z + \left(\frac{1}{2}B_1\left(c_2 - \frac{1}{2}c_1^2\right) + \frac{1}{4}B_2 c_1^2\right) z^2 + \cdots. \quad (5.7)$$

Since $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$, the Taylor series expansion of the function zf'/f is given by

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + (-a_2^2 + 2a_3)z^2 + (a_2^3 - 3a_2 a_3 + 3a_4)z^3 + (-a_2^4 + 4a_2^2 a_3 - 2a_3^2 - 4a_2 a_4 + 4a_5)z^4 + \cdots. \quad (5.8)$$

Using (5.4), (5.7) and (5.8), the coefficients a_2 and a_3 can be expressed as a function of the coefficients c_i of $p \in \mathcal{P}$ and B_i of φ as follows:

$$a_2 = \frac{1}{2}B_1 c_1 \quad (5.9)$$

and

$$a_3 = \frac{1}{8}\left((B_1^2 - B_1 + B_2)c_1^2 + 2B_1 c_2\right) \quad (5.10)$$

and

$$a_4 = \frac{1}{48}\Big((2B_1 - 3B_1^2 + B_1^3 - 4B_2 + 3B_1 B_2 + 2B_3)c_1^3 + 8B_1 c_3 + (-8B_1 + 6B_1^2 + 8B_2)c_1 c_2\Big).$$

The equations (5.9) and (5.10) readily shows that

$$a_3 - \mu a_2^2 = \frac{B_1}{4}\left(c_2 - v c_1^2\right)$$

where

$$v := \frac{1}{2}\left(1 - \frac{B_2}{B_1} + (2\mu - 1)B_1\right).$$

The desired result follows by applying Lemma 2.3. □

Corollary 5.8. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. *If* $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{CV}(\varphi)$, *then*

$$|a_3 - \mu a_2^2| \leqslant \frac{B_1}{6} \max\left\{1; \left|\frac{B_2}{B_1} + (1 - 3\mu/2)B_1\right|\right\}. \quad (5.11)$$

The result is sharp.

The previous theorem was proved with the assumption $B_1 > 0$ and univalence of φ and other properties were not used. This method is difficult to apply to get bounds for $|a_n|$ for large n, as a_n can only be expressed as a non-linear function of the coefficients c_k. As application of the Fekete-Szegő inequality for starlike functions, one can obtain the corresponding inequality for concave univalent functions [35]. For more information about concave univalent functions, see [36].

Open Problem 5.2. *Determine the sharp bound for the Taylor coefficients* $|a_n|$ $(n \geqslant 5)$ *for* $f \in \mathcal{CV}(\varphi)$ *and* $f \in \mathcal{ST}(\varphi)$. *The same problem for the other classes defined by subordination is still open.*

By expressing the first two coefficients of inverse functions of functions in $\mathcal{CV}(\varphi)$ or $\mathcal{ST}(\varphi)$ in terms of coefficients of $p_1 \in \mathcal{P}$, we can give the sharp upper bounds on the first two coefficients of inverses of functions in these two classes. We omit the details here. The sharp order of growth for the coefficients of functions in $\mathcal{CV}(\varphi)$ when $\varphi \in H^2$, the Hardy class of analytic functions in $\mathbb{D}$, is given in the following results.

Theorem 5.6. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots \in H^2$. *For* $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{CV}(\varphi)$, *we have the sharp order of growth* $|a_n| = O\left(1/n^2\right)$.

Corollary 5.9. *Let* $\varphi \in H^2$ *and* $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{ST}(\varphi)$. *Then we have the sharp order of growth* $|a_n| = O(1/n)$.

5.3 Convolution

Recall that the **convolution** of two analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

is the analytic function defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

The convolution of two functions in $\mathcal{A}$ is again in $\mathcal{A}$. Consider the class $\widetilde{\mathcal{ST}}_\alpha$ of all functions $f \in \mathcal{A}$, with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, satisfying the inequality

$$\sum_{n=2}^{\infty} (n - \alpha)|a_n| \leqslant 1 - \alpha.$$

Clearly, a function $f \in \widetilde{\mathcal{ST}}_\alpha$ is starlike of order α. If $g \in \mathcal{A}$ is convex and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, then $|b_n| \leqslant 1$. From these, it follows that

$$\sum_{n=2}^{\infty} (n - \alpha)|a_n b_n| \leqslant \sum_{n=2}^{\infty} (n - \alpha)|a_n| \leqslant 1 - \alpha$$

and so the $f * g \in \widetilde{\mathcal{ST}}_\alpha$. In other words, the class $\widetilde{\mathcal{ST}}_\alpha$ is closed under convolution with convex functions. Since the nth coefficient of normalized univalent function is bounded by n, the convolution of the Koebe function $k(z) = z/(1 - z)^2$ with itself is not univalent. Thus, the convolution of two univalent (or starlike) functions need not be univalent. Indeed, the convolution of the Koebe function κ with itself is given by

$$(\kappa * \kappa)(z) = \sum_{n=1}^{\infty} n^2 z^n = \frac{z + z^2}{(1 - z)^3}.$$

The derivative of the convolution $\kappa * \kappa$ is given by

$$(\kappa * \kappa)'(z) = \frac{1 + 4z + z^2}{(1 - z)^4}$$

and it vanishes at $z = \sqrt{3} - 2 \in \mathbb{D}$. This shows that the convolution $\kappa * \kappa$ is not even locally univalent in $\mathbb{D}$.

Pólya and Schoenberg [34] conjectured that the class of convex functions $\mathcal{CV}$ is preserved under convolution with convex functions:

$$f, g \in \mathcal{CV} \Rightarrow f * g \in \mathcal{CV}.$$

In 1973, Ruscheweyh and Sheil-Small [47] (see also [49]) proved the Polya-Schoenberg conjecture. In fact, they also proved that the classes of starlike functions and close-to-convex functions are closed under convolution with convex functions. The proof of these facts follow from the following result which is also used below to show that several other classes are closed under convolution with convex functions or starlike functions of order 1/2.

Theorem 5.7. [49, Theorem 2.4, p. 54] *Let* $\alpha \leqslant 1$, $f \in \mathcal{R}_\alpha$ *and* $g \in \mathcal{ST}(\alpha)$. *Then, for any analytic function* $H \in \mathcal{H}(\mathbb{D})$,

$$\frac{f * Hg}{f * g}(\mathbb{D}) \subset \overline{co}(H(\mathbb{D})),$$

where $\overline{co}(H(\mathbb{D}))$ *denote the closed convex hull of* $H(U)$.

Theorem 5.8 [46, Theorem 3.6, p. 131]. *Let* φ *be a convex function with* $\operatorname{Re}\varphi(z) \geqslant \alpha$, $\alpha < 1$. *If* $f \in \mathcal{R}_\alpha$ *and* $g \in \mathcal{ST}(\varphi)$, *then* $f * g \in \mathcal{ST}(\varphi)$.

The proof of this theorem follows readily from Theorem 5.7 by putting $H(z) = zg'(z)/g(z)$. In view of the fact that $f \in \mathcal{CV}(\varphi)$ if and only if $zf' \in \mathcal{ST}(\varphi)$, an immediate consequence of the above theorem is the corresponding result for $\mathcal{CV}(\varphi)$: if $f \in \mathcal{R}_\alpha$ and $g \in \mathcal{CV}(\varphi)$, then $f * g \in \mathcal{CV}(\varphi)$ for any convex function φ with $\operatorname{Re}\varphi(z) \geqslant \alpha$. For several related results for the class $\mathcal{ST}_g(\varphi)$, see [57]. For some recent results, see [28] and references therein.

It is also known [39] that the convolution $f * g$ is starlike provided the functions $f, g \in \mathcal{R}(\beta)$ of all analytic functions $f \in \mathcal{A}$ that satisfy the condition $\operatorname{Re} f'(z) > \beta$ for some $\beta > 1 - 1/(2\sqrt{1 - \ln 2}) \approx 0.097$. See [2,12,38] for similar results involving non-univalent functions. For radius problem for the class $\mathcal{R}(0)$, see MacGregor [26].

Theorem 5.9 [43]. *Let* $f \in \mathcal{A}$, $\varphi \in \mathcal{P}$ *and* $\varphi(z) = 1/q(z)$. *Then* $f \in S^*(\varphi)$ *if and only if*

$$\frac{1}{z}\left[f(z) * \left(\frac{z + z^2/(q(e^{i\theta}) - 1)}{(1 - z)^2}\right)\right] \neq 0$$

for all $z \in \mathbb{D}$ *and* $0 \leqslant \theta < 2\pi$.

Proof. Since $\frac{zf'(z)}{f(z)} \prec \varphi(z)$ if and only if

$$\frac{zf'(z)}{f(z)} \neq \varphi(e^{i\theta})$$

it follows that

$$\frac{1}{z}(zf'(z) - f(z)\varphi(e^{i\theta})) \neq 0$$

for $z \in \mathbb{D}$ and $0 \leqslant \theta < 2\pi$. Since $zf'(z) = f * \frac{z}{(1-z)^2}$ and $f(z) = f(z) * \frac{z}{1-z}$, the above inequality is equivalent to

$$\frac{1}{z}\left[f * \left(\frac{z}{(1-z)^2} - \frac{\varphi(e^{i\theta})z}{1-z}\right)\right] \neq 0,$$

which proves the result. □

Corollary 5.10 [43]. *Let $f \in \mathcal{A}$, $\varphi \in \mathcal{P}$ and $\varphi(z) = 1/q(z)$. Then $f \in C(\varphi)$ if and only if*

$$\frac{1}{z}\left[f(z) * \left(\frac{z + (1 + \frac{2}{q(e^{i\theta})-1})z^2}{(1-z)^3}\right)\right] \neq 0$$

for all $z \in \mathbb{D}$ and $0 \leqslant \theta < 2\pi$.

In particular, if $\varphi(z) = (1 + Az)/(1 + Bz), -1 \leqslant B < A \leqslant 1$, then the following results of Silverman and Silvia [53] are obtained as special cases of the previous theorems.

Corollary 5.11 [53]. *A function $f \in S^*[A, B]$ if and only if for all $z \in \mathbb{D}$ and all ζ, with $|\zeta| = 1$,*

$$\frac{1}{z}\left[f * \frac{z + \frac{\zeta - A}{A-B}z^2}{(1-z)^2}\right] \neq 0.$$

Corollary 5.12 [53]. *A function $f \in C[A, B]$ if and only if for all $z \in \mathbb{D}$ and all ζ, with $|\zeta| = 1$,*

$$\frac{1}{z}\left[f * \frac{z + \frac{2\zeta - A - B}{A-B}z^2}{(1-z)^3}\right] \neq 0.$$

5.4 Radius Problems

The assumption φ is a function with positive real part, we know that $\mathcal{ST}(\varphi)$ is a subclass of $\mathcal{ST}$. However, it may not be a subclass of $\mathcal{ST}(\beta)$. We have the following radius results [13].

Theorem 5.10. *Let $\varphi(z)$ be an analytic function with positive real part on $\mathbb{D}$ with $\varphi(0) = 1$, $\varphi'(0) > 0$ which maps the unit disk $\mathbb{D}$ onto a region starlike with respect to 1 and symmetric with respect to real axis. Let*

$$\min_{|z|=r} \text{Re } \varphi(z) = \varphi(-r). \tag{5.12}$$

Then the radius $R(\beta)$ of starlikeness of order β of functions in $ST(\varphi)$ is given by

$$R(\beta) = \begin{cases} 1 & (\varphi(-1) \geqslant \beta) \\ -\varphi^{-1}(\beta) & (\varphi(-1) \leqslant \beta). \end{cases}$$

The condition 5.12 leads to explicit expression for $R(\beta)$. If the condition is dropped $R(\beta)$ is computed such that the condition Re $\varphi(z) \geqslant \beta$ is satisfied. Also we have the following corollaries.

Corollary 5.13. *Let $f \in ST(\alpha)$, $0 \leqslant \alpha < 1$. For $0 \leqslant \beta < 1$, $f \in ST(\beta)$ in $|z| \leqslant R(\beta)$ where*

$$R(\beta) = \begin{cases} 1 & (\alpha \geqslant \beta) \\ \frac{1-\beta}{1+\beta-2\alpha} & (0 \leqslant \alpha \leqslant \beta). \end{cases}$$

Proof. Since the function f is starlike of order α we have

$$\frac{zf'(z)}{f(z)} \prec \varphi(z) = \frac{1 + (1-2\alpha)z}{1-z}.$$

Clearly we have

$$\varphi^{-1}(z) = \frac{z-1}{z+1-2\alpha}$$

and $\varphi(-1) \geqslant \beta$ is equivalent to $\alpha \geqslant \beta$. The function $\varphi(z)$ satisfies the conditions of Theorem 5.10 and therefore the results follows from Theorem 5.10. □

Corollary 5.14. *Let $f \in ST[A, B]$, $-1 \leqslant B < A \leqslant 1$. Then the function f is starlike of order β in $|z| \leqslant R(\beta)$ where*

$$R(\beta) = \begin{cases} 1 & (0 \leqslant \beta < \frac{1-A}{1-B}) \\ \frac{1-\beta}{A-\beta B} & (\frac{1-A}{1-B} \leqslant \beta < 1). \end{cases}$$

Suppose $f \in ST(\varphi)$. Then we have

$$\frac{zf'(z)}{f(z)} \prec \varphi(z), z \in \mathbb{D}.$$

If $\varphi \prec \psi$, then $f \in ST(\psi)$. Otherwise, to determine $ST(\psi)$ radius of $ST(\varphi)$ we have to find the largest $r \leqslant 1$ such that $\varphi(rz) \prec \psi(z), z \in \mathbb{D}$. This is equivalent to

$$|\psi^{-1}(\varphi(rz))| \leqslant 1, z \in \mathbb{D}. \tag{5.13}$$

Using this idea we compute the $ST[A, B]$ radius of $ST[C, D]$ in the following theorem.

Theorem 5.11. *Let $-1 \leqslant B < A \leqslant 1$ and $-1 \leqslant D < C \leqslant 1$. If $f \in ST[C, D]$ then the $ST[A, B]$ radius R is given by*

$$R = \min\left\{\frac{A-B}{C-D+|AD-BC|}; 1\right\}.$$

Proof. Let $P(z) = \frac{1+Az}{1+Bz}$ and $Q(z) = \frac{1+Cz}{1+Dz}$. Since $f \in ST[C, D]$ we have

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Cz}{1+Dz} = Q(z).$$

To determine the $ST[A, B]$ radius we have to determine R such that $0 < R \leqslant 1$ and $Q(Rz) \prec P(z)$ for $z \in \mathbb{D}$. Let $H = P^{-1}oQ$. Then we see that

$$H(z) = \frac{(C-D)z}{A-B+(AD-BC)z},$$

and

$$|H(z)| \leqslant \frac{(C-D)R}{A-B-|AD-BC|R} < 1$$

for $|z| = R \leqslant (A-B)/(C-D+|AD-BC|)$.

The result is sharp for the function f given by $f(z) = z/(1+Dz)^{(C-D)/D}$ if $D \neq 0$ and $f(z) = z\exp(Cz)$ if $D = 0$. □

Corollary 5.15 [53]. *The class* $ST[C, D]$ *is a subclass of* $ST[A, B]$ *if and only if*

$$|AD - BC| \leqslant (A-B) - (C-D).$$

The above corollary is an extension of the fact that $ST(\alpha) \subset ST(\beta)$ if and only if $\alpha \geqslant \beta$. Also Theorem 5.11 leads to corollary 5.13 and 5.14.

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The 29th International Conference on Finite or Infinite Dimensional Complex Analysis and Applications

(The 29th ICFIDCAA–2023)

Venue: Department of Mathematics, Ramanujan School of Mathematical Sciences, Pondicherry University

Dates: August 21st–25th, 2023.

About the Conference: This conference has been one of the prestigious and most significant conferences covering various topics concerning Mathematical Analysis including themajority of topics on Complex Analysis and its Applications. This conference has been held in many East Asian countries such as South Korea, Japan, China, Hong Kong, Macau, Vietnam, Thailand, India and Russia. The Academic programs of the conference shall consist of Plenary Sessions, Invited talks, Contributed talks and Short talks covering a wide range of topics on Mathematical Analysis.

Conference Homepage: `https://www.icfidcaa2023.com/`

Contact Details:

Convener: Dr. Rakesh Kumar Parmar
Email: `29icfidcaa2023@gmail.com`
Mobile: +91-9828506214

International Workshop on Geometric Function Theory (IWGFT 2023)

August 18–20, 2023.

Venue: Department of Mathematics, Indian Institute of Technology Madras, Chennai

About the Workshop: This workshop is mainly meant for young researchers who have already been working on complex analysis, interested in an inspiring branch of Complex Analysis, namely, Univalent functions, hyperbolic-type geometry, function spaces, special functions, harmonic and quasiconformal mappings, and Several complex variables. It will put special emphasis on the training of Ph.D. students, Postdoctoral fellows, and other young researchers from India. The three-day workshop will include a couple of problem discussion sessions from which participants will definitely get benefited by gaining professional knowledge and skills towards this recent trend of research in Geometric Function Theory.

Workshop Homepage: `https://sites.google.com/view/iwgft2023`

Contact Details: `iwgft2023@gmail.com`

Details of Workshop/Conferences in India

For details regarding Annual Foundation Schools, Advanced Instructional Schools, NCM Workshops, Instructional Schools for Teachers, Teacher's Enrichment Workshops
Visit: https://www.atmschools.org/

Name: The 29th International Conference on Finite or Infinite Dimensional Complex Analysis and Applications (The 29th ICFIDCAA-2023)
Date: August 21, 2023 – August 25, 2023
Venue: Pondicherry University, Pondicherry, India.
Visit: www.icfidcaa2023.com/

Name: International Conference on Linear Algebra and its Applications (ICLAA 2023, Manipal, India)
Date: December 18–21, 2023
Venue: Manipal Academy of Higher Education Manipal, Karnataka, India.
Visit: https://carams.in/events/iclaa2023/

Name: International Conference on Linear Algebra and its Applications (ICLAA 2023, Manipal, India)
Date: December 18–21, 2023
Venue: Manipal Academy of Higher Education Manipal, Karnataka, India.
Visit: https://carams.in/events/iclaa2023/

Details of Workshop/Conferences Abroad

For details regarding SIAM conferences, visit http://www.siam.org/meetings/
Name: VIASM-IAMP Summer School In Mathematical Physics
Date: August 1, 2023 – August 5, 2023
Venue: Quy Nhon University, Quy Nhon, Vietnam.
Visit: viasm.edu.vn/hdkh/iamp2023

Name: The 11th Asian Conference On Fixed Point Theory And Optimization 2023 (ACFPTO2023)
Date: August 2, 2023 – August 5, 2023
Venue: Siam Bayshore Hotel , Pattaya, Thailand.
Visit: math.sci.nu.ac.th/acfpto2023/

Name: Advanced Studies Institute In Analysis On Fractal Spaces And Dynamical Systems
Date: August 4, 2023 – August 9, 2023
Venue: Urgench State University, Urgench, Uzbekistan.
Visit: www.fullerton.edu/ires-uz/asi/asi_fractalspaces/asi_fractal-dynamical.php

Name: Women In Data Science And Mathematics (WiSDM)
Date: August 7, 2023 – August 11, 2023
Venue: Institute For Pure And Applied Mathematics (IPAM), Los Angeles, CA.
Visit: www.ipam.ucla.edu/programs/special-events-and-conferences/women-in-data-science-and-mathematics/

Name: International Algebra Conference In The Philippines
Date: August 7, 2023 – August 11, 2023
Venue: Mactan, Cebu, Philippines.
Visit: sites.google.com/g.msuiit.edu.ph/international-algebra-conferen/

Name: Finite Dimensional Integrable Systems In Geometry And Mathematical Physics (FDIS 2023)
Date: August 7, 2023 – August 11, 2023
Venue: University Of Antwerp, Belgium.

Visit: www.uantwerpen.be/nl/personeel/sonja-hohloch/private-webpage/conference-workshop/fdis2023/

Name: Groups, Actions, and Geometries Conference
Date: August 7, 2023 – August 11, 2023
Venue: Tufts University, Medford, MA.
Visit: sites.google.com/view/groups-actions-geometries/home

Name: 7th Biennial International Group Theory Conference
Date: August 7, 2023 – August 11, 2023
Venue: North-West University (Potchefstroom Campus), South Africa.
Visit: sites.google.com/view/7bigtc

Name: Quasiworld Workshop
Date: August 14, 2023 – August 18, 2023
Venue: University Of Helsinki, Finland.
Visit: www.helsinki.fi/en/conferences/quasiworld-workshop

Name: Research In Industrial Projects For Students (RIPS) Celebration 2023
Date: August 18, 2023 – August 18, 2023
Venue: Institute For Pure And Applied Mathematics (IPAM), Los Angeles, CA.
Visit: www.ipam.ucla.edu/programs/special-events-and-conferences/research-in-i ndustrial-projects-for-students-rips-celebration-2023/

Name: 54th Annual Iranian Mathematics Conference
Date: August 23, 2023 – August 25, 2023
Venue: Department Of Mathematics And Computer Sciences, Faculty Of Sciences, University Of Zanjan, Zanjan, IR Iran.
Visit: aimc54.znu.ac.ir

Name: The 6th Mediterranean International Conference of Pure & Applied Mathematics And Related Areas (MICOPAM2023)
Date: August 23, 2023 – August 27, 2023
Venue: Université D'Evry Val D'Essonne, Paris, FRANCE.
Visit: micopam.com/

Name: Connections Workshop: Algorithms, Fairness, And Equity
Date: August 24, 2023 – August 25, 2023
Venue: SLMath 17 Gauss Way, Berkeley, CA 94720.
Visit: www.msri.org/workshops/1050

Name: 5th Canada-Mexico-US Conference In Representation Theory, Noncommutative Algebra, And Categorification
Date: August 24, 2023 – August 27, 2023
Venue: Centre De Recherches Mathématiques (CRM), Montreal, Canada.
Visit: www.crmath.ca/en/activities/{#}/type/activity/id/3879

Name: 23rd International Pure Mathematics Conference 2023
Date: August 26, 2023 – August 28, 2023
Venue: Islamabad, Pakistan.
Visit: www.pmc.org.pk

Name: Complex Differential And Difference Equations II
Date: August 27, 2023 – September 2, 2023
Venue: Banach Center, Będlewo, Poland.
Visit: www.impan.pl/23-cdde2

Name: Introductory Workshop: Algorithms, Fairness, And Equity
Date: August 28, 2023 – September 1, 2023
Venue: SLMath 17 Gauss Way, Berkeley, CA 94720.
Visit: www.msri.org/workshops/1051

Name: 25th Central European Number Theory Conference
Date: August 28, 2023 – September 1, 2023
Venue: University Of Sopron, Sopron, Hungary.
Visit: `cent2023.uni-sopron.hu`

Name: XLIII Dynamics Days Europe
Date: September 3, 2023 – September 8, 2023
Venue: Scuola Superiore Meridionale, Naples, Italy.
Visit: `sites.google.com/view/dynamicsdayseurope2023`

Name: International Conference On Enumerative Combinatorics And Applications ICECA 2023 (Virtual.)
Date: September 4, 2023 – September 6, 2023
Visit: `ecajournal.haifa.ac.il/Conf/ICECA2023.html`

Name: Spanish-Polish Mathematical Meeting
Date: September 4, 2023 – September 8, 2023
Venue: University Of Lodz And Technical University Of Lodz, Poland.
Visit: `es-pl.math.uni.lodz.pl`

Name: XIII Annual International Conference of The Georgian Mathematical Union
Date: September 4, 2023 – September 9, 2023
Venue: Batumi Shota Rustaveli State University, Batumi, Georgia.
Visit: `gmu.gtu.ge/conferences/`

Name: AIM Workshop: Syzygies And Mirror Symmetry
Date: September 5, 2023 – September 8, 2023
Venue: American Institute of Mathematics, Pasadena, California.
Visit: `aimath.org/workshops/upcoming/syzygyms/`

Name: 7th IMA Conference On Mathematics In Defence And Security
Date: September 7, 2023 – September 7, 2023
Venue: Imperial College, London, UK.
Visit: `ima.org.uk/20850/7th-ima-defence/`

Name: Connections Workshop: Mathematics And Computer Science of Market And Mechanism Design
Date: September 7, 2023 – September 8, 2023
Venue: SLMath/MSRI, 17 Gauss Way Berkeley, CA 94720.
Visit: `www.msri.org/workshops/980`

Name: Connections Workshop: Mathematics and Computer Science of Market And Mechanism Design
Date: September 7, 2023 – September 8, 2023
Venue: SLMath 17 Gauss Way, Berkeley CA 94720.
Visit: `www.msri.org/workshops/980`

Name: 2023 AMS Fall Eastern Sectional Meeting, Buffalo, NY
Date: September 9, 2023 – September 10, 2023
Venue: University At Buffalo (SUNY), Buffalo, NY.
Visit: `www.ams.org/meetings/sectional/2302_program.html`

Name: Dynamics And Asymptotics In Algebra And Number Theory (DAAN)
Date: September 11, 2023 – September 15, 2023
Venue: University Of Bielefeld, Department Of Mathematics, Universitätsstrasse25, 33615 Bielefeld, Germany.
Visit: `www.math.uni-bielefeld.de/daan23/`

Name: AIM Workshop: Geometry And Topology Of Artin Groups.
Date: September 11, 2023 – September 15, 2023

Venue: Pasadena, CA.
Visit: aimath.org/workshops/upcoming/geomartingp/

Name: Mathematical And Computational Challenges In Quantum Computing
Date: September 11, 2023 – December 15, 2023
Venue: Institute For Pure And Applied Mathematics (IPAM), Los Angeles, CA.
Visit: www.ipam.ucla.edu/programs/long-programs/mathematical-and-computational-challenges-in-quantum-computing/

Name: Big Data Conference, 2023
Date: September 21, 2023 – September 23, 2023
Venue: Austin, Texas.
Visit: datascience-machinelearning.averconferences.com/

Name: AIM Workshop: Macaulay2: Expanded Functionality And Improved Efficiency
Date: September 25, 2023 – September 29, 2023
Venue: American Institute of Mathematics, Pasadena, California.
Visit: aimath.org/workshops/upcoming/macaulay2efie/

Name: 3rd International Conference On Econometrics And Business Analytics (ICEBA)
Date: September 28, 2023 – October 1, 2023
Venue: Tashkent State University Of Economics, Tashkent, Uzbekistan.
Visit: ceba-lab.org/conference23

Name: Workshop I: Quantum Algorithms For Scientific Computation
Date: October 2, 2023 – October 6, 2023
Venue: Institute For Pure And Applied Mathematics (IPAM), Los Angeles, CA.
Visit: www.ipam.ucla.edu/programs/workshops/workshop-i-quantum-algorithms-for-scientific-computation/

Name: AIM Workshop: Rigidity Properties Of Free-By-Cyclic Groups
Date: October 2, 2023 – October 6, 2023
Venue: Pasadena, CA.
Visit: aimath.org/workshops/upcoming/freebycyclic/

Name: Women In Automorphic Forms
Date: October 4, 2023 – October 6, 2023
Venue: University Of Konstanz, Universitaetsstrasse 10, 78464 Konstanz, GERMANY.
Visit: mathematik.uni-konstanz.de/women-in-automorphic-forms-2023

Name: Workshop II: Mathematical Aspects Of Quantum Learning
Date: October 16, 2023 – October 20, 2023
Venue: Institute For Pure And Applied Mathematics (IPAM), Los Angeles, CA.
Visit: www.ipam.ucla.edu/programs/workshops/workshop-ii-mathematical-aspects-of-quantum-learning/

Name: Randomization, Neutrality, And Fairness
Date: October 23, 2023 – October 27, 2023
Venue: SLMath 17 Gauss Way, Berkeley, CA 94720.
Visit: www.msri.org/workshops/1083

Name: AIM Workshop: K3: A New Problem List In Low-Dimensional Topology
Date: October 30, 2023 – November 3, 2023
Venue: Pasadena, CA.
Visit: aimath.org/workshops/upcoming/kirbylist/