Volume 34 No. 3 December 2023

ISSN 0971-1694

MATHEMATICS NEWSLETTER

SPONSORED BY :
NATIONAL BOARD FOR HIGHER MATHEMATICS

Published by
RAMANUJAN MATHEMATICAL SOCIETY

The generalized Eulerian power sums $\sum_{k=1}^{n} k^m z^k$

Raymond Mortini

Université de Lorraine, Département de Mathématiques et Institut Élie Cartan de Lorraine, CNRS, F-57000 Metz, France

Current address:

Université du Luxembourg, Département de Mathématiques, L-4364 Esch-sur-Alzette, Luxembourg

E-mail: raymond.mortini@univ-lorraine.fr

Abstract. We give a representation of the generalized Eulerian power sums $\sum_{k=1}^{n} k^m z^k$.

Keywords. Power sums, Eulerian numbers, Taylor polynomial.

AMS Subject Classification 2020: Primary 30B10, Secondary 40A05; 11B99.

1. Introduction

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}^* = \{1, 2, \dots\}$. It is a very ancient result, may be already known by Euler, that for $m \in \mathbb{N}^*$

$$E_m(z) := \sum_{k=1}^{\infty} k^m z^k = \frac{\sum_{n=1}^{m} \left[{m \atop n} \right] z^n}{(1-z)^{m+1}}, \tag{1.1}$$

where $\left[{m \atop n} \right]$ are the Eulerian numbers given by

$$\left[{m \atop n} \right] := \sum_{i=0}^{n-1} (-1)^i \binom{m+1}{i} (n-i)^m \quad (n \le m).$$

For a proof see [5], [6] as well as [1] and [7, p.143]. Note that $\left[{m \atop n} \right] \in \mathbb{N}$, as the binomial coefficients have this propertry. Let us point out that the Eulerian numbers should not to be mixed up with the Euler numbers, both are quite different classes of numbers (see [7]). Eulerian numbers also appear in combinatorics counting certain permutations (see [3] or [7, p. 144f]). Properties of the Eulerian numbers are given in [5], [3], [7] and are of course listed on wikipedia. One important one is the symmetry:

$$\left[{m \atop n} \right] = \left[{m \atop m-n+1} \right]. \tag{1.2}$$

The intention of our small note is to give for $m \in \mathbb{N}$ a similar representation of the finite Taylor sums

$$E_{n,m}(z) := \sum_{k=1}^{n} k^m z^k$$

of the functions E_m. We call these functions *generalized Eulerian power sums*, as they generalize of course the numbers $\sum_{k=1}^{n} k^m$.

On Mathstackexchange [9] it was asked for a general formula for $E_{n,m}$. For instance, as every undergraduate student of mathematics should know,

$$E_{n,0}(z) = \frac{z - z^{n+1}}{1-z}.$$

As well illustrated in [9], the explicit formulas get very difficult with increasing power m. For instance

$$E_{n,1}(z) = \frac{nz^{n+2} - (n+1)z^{n+1} + z}{(1-z)^2},$$

and

$$E_{n,2}(z) = \frac{-n^2 z^{n+3} + (2n^2 + 2n - 1)z^{n+2} - (n-1)^2 z^{n+1} + z^2 + z}{(1-z)^3}.$$

When dealing with this question, I was surprised that one could readily give such a general formula, and I posted this on [9] under my abbreviated prename "Ray". Here I present the details, hoping that the readers of these Mathematics Newsletters of the Ramanujan Mathematical society will enjoy seeing how to develop such a formula.

2. The explicit value of the generalized Eulerian sums

Our proof will be based on the method given in [5], which amounts in using the difference operator D^p. That is, let $(c_n)_{n \in \mathbb{Z}}$ be a double sided sequence of real numbers and put

$$D^0 c_n = c_n, \quad D^1 c_n = c_n - c_{n-1}$$

$$D^{p+1}c_n = D^1(D^p c_n), \quad (p \in \mathbb{N}^*).$$

It is easily seen by induction (and of course well known) that

$$D^p c_n = \sum_{j=0}^{p} (-1)^j \binom{p}{j} c_{n-j}. \tag{2.1}$$

The crux is now the following fact, which is easily proven by induction, too: If $c_{-n} = 0$ for $n \in \mathbb{N}^*$ and $\limsup_{n\to\infty} \sqrt[n]{|c_n|} \leq 1$ then

$$(1-z)^p \sum_{n=0}^{\infty} c_n z^n = \sum_{n=0}^{\infty} (D^p c_n) z^n, \quad |z| < 1. \tag{2.2}$$

We are now ready to give the value for the generalized Eulerian sums.

Theorem 2.1. *Let $m \in \mathbb{N}$. Then, for $z \in \mathbb{C}\backslash\{1\}$,*

$$\sum_{k=1}^{n} k^m z^k = \frac{1}{(1-z)^{m+1}} \times \left(\sum_{j=0}^{\max\{1,m\}} (a_j - b_{j,n}) z^j + (1 - z^{n+1}) \sum_{j=0}^{\max\{1,m\}} b_{j,n} z^j \right), \tag{2.3}$$

where $a_0 := a_0(m) := 0$,

$$a_j := a_j(m) := \sum_{i=0}^{j-1} (-1)^i \binom{m+1}{i} (j-i)^m = \left[{m \atop j} \right], \quad (j \in \mathbb{N}^*),$$

and

$$b_{j,n} := b_{j,n}(m) := \sum_{i=0}^{j} (-1)^i \binom{m+1}{i} (j-i+1+n)^m, \quad (j \in \mathbb{N}).$$

For technical reasons, and in order to better compare the a_j with the $b_{j,n}$, we replaced here $\left[{m \atop j} \right]$ by a_j (these are the Eulerian numbers). Note, though, that all the coefficients a_j and $b_{j,n}$ depend on the parameter m.

Proof. We first take $|z| < 1$. Then we may write $E_{n,m}(z)$ as

$$\sum_{k=1}^{n} k^m z^k = \sum_{k=1}^{\infty} k^m z^k - \sum_{k=n+1}^{\infty} k^m z^k$$
$$= \sum_{k=1}^{\infty} k^m z^k - z^{n+1} \sum_{k=0}^{\infty} (k+n+1)^m z^k.$$

Hence, by (2.2),

$$(1-z)^{m+1} \sum_{k=1}^{n} k^m z^k$$
$$= (1-z)^{m+1} \sum_{k=1}^{\infty} k^m z^k - z^{n+1} (1-z)^{m+1} \sum_{k=0}^{\infty} (k+n+1)^m z^k$$
$$= (1-z)^{m+1} E_m(z) - z^{n+1} \sum_{k=0}^{\infty} (D^{m+1} c_k) z^k,$$

where $c_k = \begin{cases} (k+n+1)^m & \text{if } k \in \mathbb{N} \\ 0 & \text{if } k < 0. \end{cases}$

By equation (2.1)

$$b_{k,n} := D^{m+1} c_k = \sum_{i=0}^{k} (-1)^i \binom{m+1}{i} (k-i+1+n)^m.$$

Now let $m \geq 1$. Since for positive indices c_k is a polynomial of degree m, $D^{m+1}(c_k) = 0$ for $k > m$, and by using (1.1), we conclude that

$$(1-z)^{m+1} \sum_{k=1}^{n} k^m z^k = \sum_{k=0}^{m} a_k z^k - z^{n+1} \sum_{k=0}^{m} b_{k,n} z^k, \tag{2.4}$$

from which we deduce the assertion of the theorem whenever $m \geq 1$. If $m = 0$, then

$$a_k(0) = \sum_{i=0}^{k-1} (-1)^i \binom{1}{i} (k-i)^0 = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k \geq 2. \end{cases}$$

and

$$b_{k,n}(0) = \sum_{i=0}^{k} (-1)^i \binom{1}{i} (k-i+1+n)^0 = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \geq 1. \end{cases}$$

Hence the right hand side in (2.3) equals $\frac{z - z^{n+1}}{1-z}$, which coincides with $E_{n,0}(z)$.

Thus formula (2.3) holds for $|z| < 1$. The unicity theorem for holomorphic functions now shows the validity of the formula for all $z \in \mathbb{C}$, $z \neq 1$. $\square$

Remark 2.2. Here we make the following observations (note that $a_k = a_k(m) = \left[{m \atop k} \right]$ and $b_{k,n} = b_{k,n}(m)$):

(1) By taking $z = 1$ in (2.4), we see that for every $n \in \mathbb{N}^*$ and $m \in \mathbb{N}^*$,

$$\sum_{k=0}^{m} b_{k,n}(m) = \sum_{k=0}^{m} a_k(m) = m!,$$

where the last equality comes from [5, formula (3)]. An amazing fact!

(2) Recall that for $n, m \in \mathbb{N}^*$ and $\left[{m \atop 0} \right] := 0$,

$$E_{n,m}(1) = \sum_{k=1}^{n} k^m = \sum_{j=0}^{m} \left[{m \atop j} \right] \binom{n+j}{m+1}.$$

In fact, by Wopitzky's formula (see [8], [3, p. 255] [1], and [7, p. 139]) for $m \in \mathbb{N}^*$ and $x \in \mathbb{R}$,

$$x^m = \sum_{j=0}^{m} \left[{m \atop j} \right] \binom{x+j-1}{m}.$$

Hence

$$\begin{aligned}\sum_{k=1}^{n} k^m &= \sum_{k=1}^{n}\sum_{j=0}^{m} \left[{m \atop j} \right] \binom{k+j-1}{m}\\ &= \sum_{j=0}^{m} \left[{m \atop j} \right] \sum_{k=1}^{n} \binom{k+j-1}{m}\\ &= \sum_{j=0}^{m} \left[{m \atop j} \right] \binom{n+j}{m+1}.\end{aligned}$$

A similar formula for $E_{n,m}(1)$ in terms of the Stirling numbers of the second kind is given e.g. in [2, p. 456] respectively [7, p. 212], and the standard Bernoulli formula is nicely presented in [7, p. 211].

By the way, a nice natural proof (similar to that in [4]) of Worpitzky's formula can be given as follows, by posing $\left[{m \atop j} \right] = 0$ if $m \in \mathbb{N}^*$ and $j > m$:

$$\begin{aligned}\sum_{k=1}^{\infty} k^m z^k &\overset{(1.1)}{=} \frac{\sum_{n=1}^{m} \left[{m \atop n} \right] z^n}{(1-z)^{m+1}}\\ &= \left(\sum_{n=0}^{m} \left[{m \atop n} \right] z^n\right)\left(\sum_{n=0}^{\infty} (-1)^n \binom{-m-1}{n} z^n\right)\\ &= \left(\sum_{n=0}^{\infty} \left[{m \atop n} \right] z^n\right)\left(\sum_{n=0}^{\infty} \binom{m+n}{n} z^n\right)\\ &\overset{\text{Cauchy prod.}}{=} \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} \left[{m \atop j} \right] \binom{m+n-j}{n-j}\right) z^n.\end{aligned}$$

A comparison of the coefficients and the facts that $\left[{m \atop j} \right] = 0$ for $j > m$ as well as $\left[{m \atop 0} \right] = 0$ for $m \in \mathbb{N}^*$ yields

$$n^m = \sum_{j=1}^{\min\{n,m\}} \left[{m \atop j} \right] \binom{m+n-j}{m}.$$

Now if $n < m$, then $\binom{m+n-j}{m} = 0$ for $n < j \leq m$. Hence, using (1.2),

$$\begin{aligned}n^m &= \sum_{j=1}^{m} \left[{m \atop j} \right] \binom{m+n-j}{m}\\ &= \sum_{j=1}^{m} \left[{m \atop m-j+1} \right] \binom{m+n-j}{m}\\ &\underset{k:=m-j+1}{=} \sum_{k=1}^{m} \left[{m \atop k} \right] \binom{n+k-1}{m}.\end{aligned}$$

Now consider the polynomials $p(x) = x^m$ and

$$q(x) = \sum_{k=1}^{m} \left[{m \atop k} \right] \binom{x+k-1}{m}.$$

Then p and q have degree at most m and coincide for $x = n \in \mathbb{N}^*$. Hence they are equal.

(3) By taking $z = -1$ in (2.4), we obtain for $n, m \in \mathbb{N}^*$

$$\begin{aligned}E_{n,m}(-1) &= \sum_{k=1}^{n} k^m (-1)^k\\ &= 2^{-m-1}\left(\sum_{k=0}^{m} (-1)^k \Big(a_k(m) + (-1)^n b_{k,n}(m)\Big)\right)\\ &= 2^{-m-1}\left(\sum_{k=0}^{m} (-1)^k \left(\left[{m \atop k} \right] + (-1)^n b_{k,n}(m)\right)\right).\end{aligned}$$

A further representation of $E_{n,m}(-1)$ is given in [7, p. 219] in terms of the Euler polynomials. It would be interesting to give the exact relations between the Euler polynomials and our coefficients $b_{j,n}(m)$.

Recall from Remark 2.2 that

$$\sum_{k=0}^{m} \left[{m \atop k} \right] = m!.$$

The associated alternating sum can also be computed, (see [7, p. 222]):

$$C_m := \sum_{k=0}^{m} (-1)^{k-1} \left[{m \atop k} \right] = 2^{m+1}\,(2^{m+1}-1)\frac{B_{m+1}}{m+1},$$

[1] Attention: the symbol $\left\langle {n \atop p} \right\rangle$ in [3] corresponds to our $\left[{n \atop p+1} \right]$.

where B_n is the n-th Bernoulli number, defined to be $f^{(n)}(0)$ for the holomorphic function

$$f(z) = \begin{cases} \frac{z}{e^z-1} & \text{if } 0 < |z| < 2\pi \\ 1 & \text{if } z = 0. \end{cases}$$

In particular, $C_m = 0$ if m is even. This last assertion also follows from the symmetry of the Eulerian numbers $\left[{m \atop k}\right]$. Usually, the numbers $T_n := |C_{2n-1}|$ are called the *tangent numbers*, because

$$\sum_{n=1}^{\infty} T_n \frac{x^{2n-1}}{(2n-1)!} = \tan x, \quad (|x| < \pi).$$

We therefore call C_m the *signed tangent number*. It is easy to see that $C_m \in \mathbb{N}$ (since the Eulerian coefficients belong to $\mathbb{N}$).

Here are the first Eulerian, Bernoulli and tangent numbers:

$\diagdown$	$\left[{m \atop 1}\right]$	$\left[{m \atop 2}\right]$	$\left[{m \atop 3}\right]$	$\left[{m \atop 4}\right]$	$\left[{m \atop 5}\right]$	$\left[{m \atop 6}\right]$	$\left[{m \atop 7}\right]$
m=1	1						
m=2	1	1					
m=3	1	4	1				
m=4	1	11	11	1			
m=5	1	26	66	26	1		
m=6	1	57	302	302	57	1	
m=7	1	120	1191	2416	1191	120	1
$\vdots$							$\ddots$

$$B_0 = 1 \quad B_1 = -\tfrac{1}{2} \quad B_2 = \tfrac{1}{6} \quad B_4 = -\tfrac{1}{30}$$

$$B_6 = \tfrac{1}{42} \quad B_8 = -\tfrac{1}{30} \quad B_{10} = \tfrac{5}{66} \quad B_{12} = -\tfrac{691}{2730}$$

$$B_{14} = \tfrac{7}{6} \quad B_{16} = -\tfrac{3617}{510} \quad B_{18} = \tfrac{43867}{798} \quad B_{20} = -\tfrac{174\,611}{330}$$

$$B_{22} = \tfrac{854\,513}{138} \quad B_{24} = -\tfrac{236\,364\,091}{2730} \cdots$$

Note that $B_{2n+1} = 0$ for all $n \in \mathbb{N}^*$. And finally, we conclude this remark by giving the first signed tangent numbers:

$$C_1 = 1 \quad C_3 = -2 \quad C_5 = 16 \quad C_7 = -272 \quad C_9 = 7\,936$$

$$C_{11} = -353\,792 \quad C_{13} = 22\,368\,256$$

$$C_{15} = -1\,903\,757\,312 \quad \cdots$$

Remark 2.3. Instead of the finite differences calculus we applied here to obtain our formula 2.3, one may also use the differential operator $\left(z\frac{d}{dz}\right)^m$ to obtain a formula for $\sum_{k=1}^{n} k^m z^k$. This is based though on an inductive argument, necessitating an a priori knowledge of the formula. Its proof is lengthier. So I find our approach here more natural. This differential calculus approach to $E_m(z) = \sum_{k=1}^{\infty} k^m z^k$ was done for instance in [6].

References

[1] K. N. Boyadzhiev, Close encounters with the Stirling numbers of the second kind, *Math. Mag.*, **85** (2012), 252–266. **1**

[2] H. W. Gould, Euler's formula for n-th differences of powers, *Amer. Math. Monthly*, **85** (1978), 450–467. **5**

[3] R. L. Graham, D. E. Knuth and O. Patashnik, Concrete Mathematics, Addison-Wesley Pub. Company, (1989). **1,4**

[4] H. K. Krishnapriyan, Eulerian polynomials and Faulhaber's result on sums of powers of integers, *The College Math. Journal*, **26** (1995), 118–123. **5**

[5] R. Mortini, Die Potenzreihe $\sum n^m z^n, m \in \mathbb{N}$. *Elem. Math*, **83** (1983), 49–51. **1, 2, 4**

[6] A. Pfluger, Bemerkung zum Beitrag von R. Mortini, *Elem. Math*, **83** (1983), 102–103. **1, 7**

[7] J. Quaintance and H. W. Gould, Combinatorial Identities for Stirling Numbers, The Unpublished Notes of H W Gould, World Scientific, New Jersey, (2016). **1, 4, 5, 6**

[8] J. Worpitzky, Studien über die Bernoullischen und Eulerschen Zahlen, *J. Reine Angew. Math. – Crelle's Journal – Kronecker J.*, **94** (1883), 203–233. **4**

[9] General formula for the power sum $\sum_{k=1}^{n} k^m x^k$, $\forall m \in \mathbb{N}$. `https://math.stackexchange.com/questions/4276236/general-formula-for-the-power-sum-sum-k-1nkmxk-quad-forall-m-in-mat` **2**

A review on the closed graph theorem and the open mapping theorem via finite dimensionality

Uday Shankar Chakraborty

Department of Mathematics, Assam University, Silchar, Assam, India

E-mail: udayhkd@gmail.com

Abstract. In this note, we provide characterizations of finite dimensionality of normed linear spaces with the help of conditions which are related to the closed graph theorem and the open mapping theorem.

Keywords. Closed graph theorem, Open mapping theorem, Finite dimensionality.

AMS Subject Classification 2020: 46A30, 46B99.

1. Introduction

The closed graph theorem and the open mapping theorem are two fundamental theorems that students learn in a first course of functional analysis. These two theorems together with the Hahn-Banach theorem and the uniform boundedness principle are commonly regarded as the four pillars in functional analysis. The closed graph theorem basically gives a sufficient condition for a linear map with a closed graph to be continuous and the open mapping theorem provides a sufficient condition for a continuous surjective linear map to be open. It is interesting to see that the conclusion of both the theorems hold without any additional condition for linear functionals, i.e. for every normed linear space X, a linear functional is continuous if its graph is closed and a continuous surjective linear functional is always open. In this note, we present the fact that this is not restricted to linear functionals only. The conclusion of both the above mentioned theorems hold for any domain space if and only if the codomain space is finite dimensional. We do not provide the detailed proofs of the known results which are available in standard text books like [1].

Throughout this note, as far as possible, we stick to standard notations only. X, Y denote normed linear spaces over the field $\mathbb{K}(\mathbb{R}$ or $\mathbb{C})$. $\mathcal{L}(X, Y)$ denotes the set of all linear maps from X into Y. Given any $T \in \mathcal{L}(X, Y)$, $\ker T = \{x \in X : T(x) = 0\}$ and $\mathrm{Gr}(T) = \{(x, T(x)) : x \in X\}$ are the kernel and graph of T respectively.

2. Main Results

Definition 2.1. *A pair of normed linear spaces (X, Y) has the closed kernel property if for each $T \in \mathcal{L}(X, Y)$, T is continuous whenever* $\ker T$ *is closed in X.*

Example 2.2. *Let X be a normed linear space and let $f : X \to \mathbb{K}$ be a linear functional such that* $\ker f$ *is closed in X. If possible, let f be not continuous. Thus there exists $x_0 \in X$ such that $f(x_0) \neq 0$. Since f is not continuous at 0, there exists $\varepsilon > 0$ such that for each $n \in \mathbb{N}$, there exists $x_n \in X$ satisfying $\|x_n\| < \frac{1}{n}$ but $|f(x_n)| > \varepsilon$. Define $y_n = x_0 - \frac{f(x_0)}{f(x_n)} x_n$ for each $n \in \mathbb{N}$. Then (y_n) is a sequence in* $\ker f$ *such that $y_n \to x_0$. Since* $\ker f$ *is closed in X, we must have $x_0 \in \ker f$ entailing $f(x_0) = 0$, which is a contradiction. Hence the pair $(X, \mathbb{K})$ has the closed kernel property.*

We have plenty of examples to show that there are normed linear spaces Y for which we can find normed linear spaces X such that (X, Y) fails the closed kernel property. The question is can we find some conditions on Y under which (X, Y) satisfies the closed kernel property for every normed linear space X? Is completeness of Y the desired condition? The answer is unfortunately negative. In fact there exist normed linear spaces X such that completeness of Y does not guarantee the closed kernel property for the pair (X, Y) even for surjective linear maps. In order to illustrate our claim with the help of an example, we need the following result.

Definition 2.3. *A normed linear space X is separable if it has a countable dense subset.*

Proposition 2.4. *Every infinite dimensional separable Banach space X has vector space dimension $\mathfrak{c}$.*

Proof. By Baire category theorem, vector space dimension of X can not be less than $\mathfrak{c}$. Again, since X is separable, it has a countable dense subset, say A. Thus every $x \in X$ is the limit of a convergent sequence in A. Since the cardinality of the set of all sequences in A is $|\mathbb{N}|^{|\mathbb{N}|} = \mathfrak{c}$, therefore cardinality of X is less than equal to $\mathfrak{c}$. □

Example 2.5. *Consider the spaces ℓ^1 and ℓ^2. Both of them are separable and so they have vector space bases of dimension $\mathfrak{c}$ and so they are isomorphic as vector spaces. Suppose $T : \ell^1 \to \ell^2$ is a vector space isomorphism. Then $\ker T = \{0\}$ and so it is closed in ℓ^1. If possible, let T be continuous. Then by bounded inverse theorem, $T^{-1} : \ell^2 \to \ell^1$ is continuous and as a result ℓ^1 and ℓ^2 will be topologically isomorphic, which is a contradiction. Hence (ℓ^1, ℓ^2) does not have the closed kernel property.*

Definition 2.6. *A pair of normed linear spaces (X, Y) has*

(a) the closed graph property if for every linear map $T : X \to Y$, $Gr(T)$ is closed in $X \times Y$ implies T is continuous,

(b) the open mapping property if every continuous surjective linear map $T : X \to Y$ is open.

Example 2.7. *Let X be any normed linear space. Suppose $f : X \to \mathbb{K}$ is a linear functional such that $Gr(f)$ is closed in X. Let (x_n) be a sequence in $\ker f$ such that $x_n \to x$ in X. Thus $(x_n, f(x_n)) = (x_n, 0) \to (x, 0)$ and so $(x, 0) \in Gr(f)$ as $Gr(f)$ is closed. It follows that $f(x) = 0$ and so $x \in \ker f$. Consequently, $\ker f$ is closed entailing f to be continuous by Example 2.2. Hence $(X, \mathbb{K})$ has the closed graph property.*

Example 2.8. *Let X be any normed linear space. Suppose $f : X \to \mathbb{K}$ is a non-trivial linear functional. We claim that $f(B_1^X(0))$ contains an open ball centred at 0 in $\mathbb{K}$. Let $\delta(\neq 0) \in f(B_1^X(0))$. Then there exists $x(\neq 0) \in B_1^X(0)$ such that $f(x) = \delta$. Let $\alpha \in B_{|\delta|}^{\mathbb{K}}(0)$. Thus $|\alpha| < |\delta|$. Let $y = \frac{\alpha x}{f(x)}$. This gives $f(y) = \alpha$ and $\|y\| = \frac{|\alpha|}{|f(x)|}\|x\| < 1$. Thus $f(y) = \alpha \in f(B_1^X(0))$. Therefore, $B_{|\delta|}^{\mathbb{K}}(0) \subset f(B_1^X(0))$. Let G be any open subset of X and let $f(z) \in f(G)$. Thus there exists $r > 0$ such that $B_r^X(z) \subset G$. This implies $f(B_r(z)) \subset f(G)$. Now*

$$\begin{aligned} B_{r|\delta|}^{\mathbb{K}}(f(z)) &= f(z) + rB_{|\delta|}^{\mathbb{K}}(0) \subset f(z) + rf(B_1^X(0)) \\ &= f(B_r^X(z)) \subset f(G). \end{aligned}$$

Consequently, $f(G)$ is open in $\mathbb{K}$. Hence $(X, \mathbb{K})$ has the open mapping property.

Theorem 2.9. *(a)* ***(The Closed Graph Theorem)*** *If X and Y are Banach spaces, then the pair (X, Y) has the closed graph property.*

(b) ***(The open mapping theorem)*** *If X and Y are Banach spaces, then the pair (X, Y) has the open mapping property.*

It is possible to find examples of normed linear spaces X, Y such that the pair (X, Y) fails to have both above mentioned properties if either of X and Y is incomplete. With this we are motivated enough to ask the following questions:

Question 2.1. *Let X be a normed linear space. Under what condition on a normed linear space Y, the pair (X, Y) has*

1. *the closed graph property?*
2. *the open mapping property?*

We answer all the questions with the following result.

Proposition 2.10. *Let Y be a normed linear space. Then the following statements are equivalent.*

(a) Y is finite dimensional.

(b) (X, Y) has the closed kernel property for every normed linear space X.

(c) (X, Y) has the closed graph property for every normed linear space X.

(d) (X, Y) has the open mapping property for every normed linear space X.

Proof. (a) $\Longrightarrow$ (b): Let X be a normed linear space and let $T \in \mathcal{L}(X, Y)$ be such that $\ker T$ is closed in X. We know that $T(X)$ is isomorphic to $X/\ker T$ as vector spaces via vector space isomorphism $\tilde{T} : X/\ker T \to T(X)$ given by $\tilde{T}(x + \ker T) = T(x)$ for all $x \in X$. As Y is finite dimensional, $T(X)$ is also finite dimensional and hence so is $X/\ker T$. Thus $\tilde{T}$ is continuous. Therefore for all $x \in X$, $\|T(x)\| = \|\tilde{T}(x + \ker T)\| \leq \|\tilde{T}\|\,\|x + \ker T\| \leq \|\tilde{T}\|\,\|x\|$. Thus T is continuous. Hence (X, Y) has the closed kernel property.

(b) $\implies$ (c): It follows directly from the fact that for every $T \in \mathcal{L}(X, Y)$, closedness of $\mathrm{Gr}(T)$ implies closedness of $\ker T$.

(c) $\implies$ (d): Let X be any normed linear space and let $T \in \mathcal{L}(X, Y)$ be a continuous, surjective mapping. Continuity of T implies that $\ker T$ is closed. Clearly, the quotient map $\pi : X \to X/\ker T$ is continuous and surjective. Since for every $x \in X$ and for every $r > 0$, $\pi(B_r(x)) = B_r(\pi(x))$, therefore π is an open map. By the definition of quotient topology, there exists a continuous, linear bijection $\tilde{T} : X/\ker T \to Y$ such that $T = \tilde{T} \circ \pi$. Since $\tilde{T}$ is continuous, $\mathrm{Gr}(\tilde{T})$ is closed in $X/\ker T \times Y$. It is not difficult to show that the map

$$\varphi : X/\ker T \times Y \to Y \times X/\ker T; (x + \ker T, y) \mapsto (y, x + \ker T)$$

is a homeomorphism. Thus $\mathrm{Gr}(\tilde{T}^{-1}) = \varphi(\mathrm{Gr}(\tilde{T}))$ is closed in $Y \times X/\ker T$. By hypothesis, $\tilde{T}^{-1}$ is continuous and so $\tilde{T}$ is open. Let G be an open set in X. Thus $\pi(G)$ is open in $X/\ker T$. Consequently, $T(G) = \tilde{T}(\pi(G))$ is open in Y entailing T to be an open map. Hence (X, Y) has the open mapping property.

(d) $\implies$ (a): If possible, let Y be infinite dimensional. Then there exists a discontinuous linear map $f : Y \to \mathbb{K}$. If $\|x\|_0 = \|x\| + |f(x)|$ for all $x \in Y$, then $\|.\|_0$ is a norm on Y and $f : (Y, \|.\|_0) \to \mathbb{K}$ is continuous. Since the identity map I from $(Y, \|.\|_0)$ onto the normed linear space $(Y, \|.\|)$ is a continuous linear map, by hypothesis, I is open. Since I is one-one, $I^{-1} : (Y, \|.\|) \to (Y, \|.\|_0)$ exists and is continuous. Thus there exists $M > 0$ such that $\|x\|_0 \leq M\|x\|$ for all $x \in Y$. Thus for all $x \in Y$,

$$|f(x)| \leq \|x\|_0 \leq M\|x\|.$$

This is a contradiction as f is discontinuous.

Remark 2.11. The idea of the implication (c) $\implies$ (d) is borrowed from [2]. The implication (d) $\implies$ (a) is taken from [3].

Acknowledgement

The author expresses his thanks to the anonymous reviewer for his valuable comments and suggestions which definitely have improved the presentation of the paper.

References

[1] R. E. Megginson, An introduction to Banach space theory, Springer, New York (1998).

[2] Available online: https://math.stackexchange.com/a/90266

[3] A. K. Chakrabarty, personal communications.

The Andersén-Lempert theory on $\mathbb{C}^n$ and some of its applications

Sanjoy Chatterjee[1] and Sushil Gorai[2,*]

[1,2]*Department of Mathematics and Statistics, Indian Institute of Science Education and Research Kolkata, Mohanpur 741 246.*

E-mail: [1]sc16ip002@iiserkol.ac.in, [2]sushil.gorai@iiserkol.ac.in

Abstract. This article presents a brief survey about the Andersén-Lempert theory on $\mathbb{C}^n$. We also survey the results about the approximation of biholomorphic maps of certain domains in $\mathbb{C}^n$. A couple of applications are also provided.

Keywords. Automorphisms of $\mathbb{C}^n$, holomorphic vector fields, Fatou–Bieberbach domain.

Mathematics Subject Classification 2020: Primary: 32M17, 32E30.

1. Introduction

Let $\Omega \subseteq \mathbb{C}^n$ be a domain. A holomorphic map $f : \Omega \to \Omega$ is said to be an automorphism if f is bijective and the inverse

*Sanjoy Chatterjee is supported by CSIR fellowship (File No-09/921(0283)/2019-EMR-I). Sushil Gorai is partially supported by a Core Research Grant (CRG/2022/003560) of SERB, Govt. of India

is also holomorphic. The collection of all biholomorphisms from Ω to itself is denoted by $\mathrm{Aut}(\Omega)$. $\mathrm{Aut}(\Omega)$ forms a group under the composition of mappings. For $\Omega = \mathbb{C}$ the automorphism group $\mathrm{Aut}(\mathbb{C}) = \{z \mapsto az + b : a, b \in \mathbb{C}, a \neq 0\}$. However, For $n \geq 2$, the group $\mathrm{Aut}(\mathbb{C}^n)$ is huge. The following mappings are elementary type automorphism of $\mathbb{C}^n$: For any $f, g \in \mathcal{O}(\mathbb{C}^{n-1})$

$$z \mapsto (z_1 + f(z_2, \ldots, z_n), z_2, z_3, \ldots, z_n) \tag{1.1}$$

$$z \mapsto (e^{g(z_2,\ldots,z_n)} z_1, z_2, z_3, \ldots, z_n). \tag{1.2}$$

We also define $\mathrm{Aut}_1(\mathbb{C}^n) := \{f \in \mathrm{Aut}(\mathbb{C}^n) : \det DF(z) \equiv 1\}$. Following the terminology of Rosay-Rudin [10], the maps of type (1.1) and their $Sl(n, \mathbb{C})$ conjugates are called the shear maps, and the maps of type (1.2) and their $Gl(n, \mathbb{C})$ conjugates are called overshears. In the seminal work of Rosay and Rudin [10], the authors investigated various properties pertaining to the automorphisms of $\mathbb{C}^n$. Particularly, noteworthy is their examination of a specific class of automorphisms known as 'shears'. This elegant study sheds light on the fascinating characteristics exhibited by such automorphisms within $\mathbb{C}^n$.

It follows from [10] that the group $\mathrm{Aut}(\mathbb{C}^n)$ act m-transitively on $\mathbb{C}^n$, i.e., for every pair of m-tuples $\{a_j\}_1^m$, $\{b_j\}_1^m$ subset of $\mathbb{C}^n$ of distinct elements, there exists $g \in \mathrm{Aut}(\mathbb{C}^n)$ such that $g(a_j) = b_j$ for $j = 1, \ldots, m$. The topology in $\mathrm{Aut}(\mathbb{C}^n)$ is the compact open topology, i.e., the sense of convergence is the uniform convergence on compact subsets of $\mathbb{C}^n$.

The Andersén-Lempert theory has been generalized in the case of manifolds–now they are called the manifolds with density property. In this article, we focus on $\mathbb{C}^n$ and explain in details in Section 2. In Section 3 we give a survey of the results about approximation of biholomorphic maps by automorphisms of $\mathbb{C}^n$ and some of their applications.

2. The Andersén-Lempert theory in $\mathbb{C}^n$

In the same paper, Rosay and Rudin [10, Question 8] asked the following question: *Can every $F \in Aut(\mathbb{C}^n)$ with $JF \equiv 1$ be approximated by finite compositions of shears?*

In 1990, Andersén [1] provided an affirmative answer.

The following provides a coordinate-independent representation for both the Shear map and the Overshear map.

Let $k < n$ and $\Lambda : \mathbb{C}^n \to \mathbb{C}^k$ be a $\mathbb{C}$ linear map. Let $v \in \mathrm{Ker}\,\Lambda$ and $f \in \mathcal{O}(\mathbb{C}^k)$. Then, the following types of maps are contained in the automorphism group of $\mathbb{C}^n$: For every $t \in \mathbb{C}$

$$\text{shear: } \Phi_t(z) \mapsto z + tf(\Lambda(z))v \tag{2.1}$$

$$\text{overshear: } \Psi_t(z) \mapsto z + \frac{1}{\|v\|^2}(e^{t(\|v\|^2 f(\Lambda(z)))} - 1)\langle z, v\rangle v \tag{2.2}$$

$$z \mapsto (z_1 e^{c_1\phi(z^r)}, \ldots, z_n e^{c_n\phi(z^r)}). \tag{2.3}$$

Here $\phi\colon \mathbb{C} \to \mathbb{C}$ is a holomorphic function and $r = (r_1, r_2, \ldots, r_n) \in \mathbb{N}^n$ with $\sum_{j=1}^n c_j r_j = 0$ and $z^r := z_1^{r_1} z_2^{r_2} \ldots z_n^{r_n}$. For $1 \leq k \leq n$, let S_k^n, M_k^n be the sets consisting of automorphism of type (2.1), (2.2) respectively, where k is determined by $f \in \mathcal{O}(\mathbb{C}^k)$. For $A \subset \mathrm{Aut}(\mathbb{C}^n)$ let $G(A)$ denote the group generated by A in $\mathrm{Aut}(\mathbb{C}^n)$.

Since the maps Λ and f are holomorphic mappings, hence, Φ_t is holomorphic from $\mathbb{C}^n$ to $\mathbb{C}^n$. We have the following for every $t \in \mathbb{C}$:

$$\Phi_t(z - tf(\Lambda z)v) = z - tf(\Lambda z)v + tf(\Lambda(z - tf(\Lambda z)v))v.$$

Since Λ is a linear map and $v \in \mathrm{Ker}(\Lambda)$, hence, we get that

$$\begin{aligned}\Phi_t(z - tf(\Lambda z)v) &= z - tf(\Lambda z)v + tf(\Lambda z)v \\ &= z.\end{aligned}$$

Therefore, the inverse of the map Φ_t is $\Phi_{-t} = z - tf(\Lambda(z))v$. In a similar way, we can find that the inverse of the map Ψ_t is Ψ_{-t}.

The following lemma [6, Lemma 4.1.1] proves that the maps of type (2.1) and (2.2) are obtained by applying linear change of coordinates of the maps of the form (1.1) and (1.2) respectively.

Lemma 2.1. *Suppose Φ_t and Ψ_t are as given by (2.1) and (2.2) respectively. Then, we have:*

i. *If $A \in GL_n(\mathbb{C})$ is such that $Av = e_n$, then $A \circ \Phi_t(z) = \widetilde{\Phi}_t(Az)$, where $\widetilde{\Phi}_t$ is given by (2.1) with f replaced by $f \circ \lambda \circ A^{-1}$.*
ii. *If $A \in U(n)$ is such that $Av = |v|e_n$, then $A \circ \Psi_t(z) = \widetilde{\Psi}_t(Az)$, where $\widetilde{\Psi}_t$ is given by (2.2) with f replaced by $|v|^2 f \circ \lambda \circ A^{-1}$.*

The following result is due to Andersén [1, Theorem C].

Theorem 2.2 (Andersén). *$\overline{G(S_1^n)} = Aut_1(\mathbb{C}^n)$, where the closure is taken in compact open topology.*

In [1, Theorem B], it is also proved that

Theorem 2.3 (Andersén). *The automorphism $G : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $(z, w) \mapsto (ze^{zw}, we^{-zw})$ can not be written as a finite composition of the shear maps.*

The following result is due to Andersén-Lempert [2, Theorem 1.3].

Theorem 2.4 (Andersén-Lempert). *$G(M_1^n \cup S_1^k)$ is dense in $Aut(\mathbb{C}^n)$ with respect to compact open topology.*

Breakthroughs in [2] not only answered Rudin's question but also accelerated the pace of research in the field of $\mathrm{Aut}(\mathbb{C}^n)$.

In this article, we present a proof of Theorem 2.4. The proof of Theorem 2.4 relies heavily on the next three subsequent lemmas concerning solutions of ODEs. Most of the material is taken from [6].

Definition 2.5. *A holomorphic vector field V on $\mathbb{C}^n$ is a real vector field on $\mathbb{R}^{2n}$ of the form*

$$V(z) = \sum_{i=1}^{n} a_i(z)\frac{\partial}{\partial x_i} + b_i(z)\frac{\partial}{\partial y_i}$$

such that $(a_j + ib_j)$ is a holomorphic function for all $j \in \{1, 2, \ldots, n\}$.

The next result will be used in the proof of Proposition 2.13.

Result 2.6 ([6]). *Let V be a time-dependent continuous vector field on a domain $\Omega \subset R^{1+n}$ satisfying a uniform Lipschitz estimate*

$$|V_t(x) - V_t(y)| \leq B|x - y|,$$

for some $B > 0$. Then for any $s \in \mathbb{R}$ and any pair of points $x, y \in \Omega_s$ we have

$$\left|\phi_{t,s}(x) - \phi_{t,s}(y)\right| \leq e^{B|t-s|}|x - y|,$$

for all t such that the trajectories exist and remain in the domain $\Omega_t = \{x \in \mathbb{R}^n : (t, x) \in \Omega\}$.

The next result [4, Lemma 2.8] will play a crucial role in establishing Theorem 2.4. Specifically, it asserts that every divergence-free vector field generates volume preserving flow.

Result 2.7. *Let $\Omega \subset \mathbb{C}^n$ be open, $F : \Omega \to \mathbb{C}^n$ be a complete $\mathcal{C}^1$ vector field, and $X(t, z)$ be its flow. Then*

$$\frac{d}{dt} detD_z(X(t, z)) = div(F(X(t, z))) \cdot detD_z(X(t, z)).$$

Here $divF := \frac{\partial F_1}{\partial z_1} + \frac{\partial F_2}{\partial z_2} + \cdots + \frac{\partial F_n}{\partial z_n}$.

We will work with a time-dependent vector field $X : \mathbb{R} \times \mathbb{C}^n \to \mathbb{C}^n$. Here X is a $\mathcal{C}^1$ map and X_t is holomorphic for every fixed $t \in \mathbb{R}$. Recall that for such vector fields, the flow of X at $(t_0, z_0) \in \mathbb{R} \times \mathbb{C}^n$ is the map $\phi(\cdot; t_0, z_0)$ defined on a neighborhood I_{t_0} of t_0 which satisfies the following Cauchy problem:

$$\begin{aligned} \frac{d\phi(t; t_0, z)}{dt} &= X_t(\phi(t; t_0, z)) \\ \phi(t_0; t_0, z_0) &= z_0, \end{aligned} \tag{2.4}$$

with $t \in I_{t_0}$. We will also refer to ϕ as the solution to the vector field X. The time-dependent vector field X is said to be complete if the solution of (2.4) exists for all $(t_0, z) \in \mathbb{R} \times \mathbb{C}^n$ for all $t \in \mathbb{R}$.

Result 2.8 ([6, Page-36]). *Let V be a vector field on $\mathbb{C}^n$ for some $n \in \mathbb{N}$ and $\phi(t, x)$ be its flow. For any $z \in \mathbb{C}^n$ and any t for which $\phi_t(z)$ exists, it holds that $\phi_t(z) = z + tV(z) + o(t)$.*

Lemma 2.9. *Let $V, W \in \mathfrak{X}_{\mathcal{O}}(\mathbb{C}^n)$ be two vector fields and ϕ_t and ψ_t be the corresponding flows. Let $K_t(z) = (\phi_t \circ \psi_t)(z)$, when RHS is well defined. Then we have the following:*

i. $\frac{\partial K_t(z)}{\partial t}|_{t=0} = V(z) + W(z)$

ii. $K_t(z) - z - t(V(z) + W(z)) = o(t)$.

Proof. From Result 2.8, we get that

$$\begin{aligned} \phi_t(z) &= z + tV(z) + O(t^2) \\ K_t(z) &= \phi_t(\psi_t(z)) = \psi_t(z) + tV(\psi_t(z)) + O(t^2) \\ \frac{\partial K_t}{\partial t} &= W(\psi_t(z)) + V(\psi_t(z)) \\ &\quad + tDV(\psi_t(z))W(\psi_t)(z) + O(t). \end{aligned}$$

Taking limit $t \to 0^+$ we get that $\lim_{t\to 0} \frac{\partial K_t}{\partial t} = V(z) + W(z)$.

This proves (i).

Again from Result 2.8, we get the following:

$$\phi_t(z) = z + tV(z) + O(t^2),$$

hence, we have

$$K_t(z) = \phi_t(\psi_t(z)) = \psi_t(z) + tV(\psi_t(z)) + O(t^2),$$

therefore, from Result 2.8, we get that

$$K_t(z) = z + t(W(z) + V(\psi_t(z))) + O(t^2),$$

$$K_t(z) - z - t(V(z) + W(z)) = t(V(\psi_t(z)) - V(z)) + O(t^2),$$

$$\frac{K_t(z) - z - t(V(z) + W(z))}{t} = (V(\psi_t(z)) - V(z)) + O(t).$$

The right-hand side of the above equation goes to 0 as $t \to 0^+$. This proves the lemma. □

The following lemma is from [1].

Lemma 2.10. *Let $k, n \in N := \mathbb{N} \cup \{0\}$ and let $m_k \in \mathbb{N}$ be the cardinality of the set of multi-indices $\{I \in \mathbb{N}^n : |I| = k\}$. Then there exist linear maps $\{\lambda_j\}_{j=1}^{m_k}$, $\lambda_j : \mathbb{C}^n \to \mathbb{C}$, such that any homogeneous polynomial P of degree k is of the form*

$$P(z) = \sum_{j=1}^{m_k} c_j (\lambda_j(z))^k, \quad c_j \in \mathbb{C}.$$

Proof. We can always write $\lambda_j(z) = \langle z, \overline{a}_j \rangle$ for a certain $a_j \in \mathbb{C}^n$, $j \in \{1, \dots, m_k\}$. Then $(\lambda_j(z))^k = \sum_{|I|=k} \binom{k}{I} a_j^I z^I$, where $z^I := z_1^{I_1} z_2^{I_2} \dots z_n^{I_n}$ and $\binom{k}{I} := \frac{k!}{I_1! I_2! \dots I_k!}$. Now we have the following:

$$\sum_{j=1}^{m_k} c_j (\lambda_j(z))^k = \sum_{j=1}^{m_k} c_j \sum_{|I|=k} \binom{k}{I} a_j^I z^I = \sum_{|I|=k} \binom{k}{I} z^I \sum_{j=1}^{m_k} c_j a_j^I.$$

Since $P(z)$ is homogeneous polynomial of degree k,

$$P(z) = \sum_{|I|=k} P_I z^I.$$

It is enough to find a solution

$$\binom{k}{I} \sum_{j=1}^{m_k} c_j a_j^I = P_I.$$

Consider the $m_k \times m_k$ matrix $A = (a_j^I)_{j=1,2,\dots,m_k}^{|I|=k}$ where $|I| = k$ for $j \in \{1, \dots, m_k\}$. The previous equation has a solution for $\det A \neq 0$. Choose the entries of the vectors a_j to be multiplicatively independent over $\mathbb{Q}$, i.e., $a_{j_1}^{I_1} = a_{j_2}^{I_2}$ if and only if $j_1 = j_2$ and $I_1 = I_2$. With this choice, $\det A$ is a Vandermonde determinant and it is not zero. □

Remark 2.11. Here $\det A$ is a homogeneous polynomial of degree km in mn variables, and the set $\{\det A \neq 0\}$ is a non-empty open dense set of $\mathbb{C}^{mn}$. Hence, given any open set $U \subset (\mathbb{C}^n)^*$ (the dual of $\mathbb{C}^n$), we can choose our linear maps to be in U.

Lemma 2.12 ([11, Theorem 2.8]). *Let $\Omega \subset \mathbb{C}^n$ be a domain, $I \subset \mathbb{R}$ an open interval, and $f, g : I \times \mathbb{C}^n \to \mathbb{C}^n$ be time-dependent continuous vector fields. Suppose f is locally Lipschitz on the second variable, uniformly with respect to the first one. Let ψ and ϕ be the solutions to the following Cauchy problems:*

$$\begin{cases} \frac{d\psi(t;t_0,w)}{dt} = g(t, \psi(t; t_0, w)) \\ \psi(t_0; t_0, w_0) = w_0 \end{cases} \qquad \begin{cases} \frac{d\phi(t;t_0,z)}{dt} = f(t, \phi(t; t_0, z)) \\ \phi(t_0; t_0, z_0) = z_0. \end{cases}$$

Then we have that

$$\|\phi(t; t_0, z_0) - \psi(t; t_0, w_0)\| \leq \|z_0 - w_0\| e^{L|t-t_0|} + M \cdot \frac{e^{L|t-t_0|} - 1}{L}$$

where

$$L = \sup_{(t,z) \neq (t,w) \in U} \frac{\|f(t, z) - f(t, w)\|}{\|z - w\|},$$

and

$$M = \sup_{(t,z) \in U} \|f(t, z) - g(t, z)\|$$

with U being a set containing both graphs of $\phi(t; t_0, z_0)$ and $\psi(t; t_0, w_0)$.

The following proposition outlines a method for approximating the flow of a vector field, which is expressed as a finite sum of complete vector fields. This implies that if a vector field defined on $\mathbb{C}^n$ is represented as a finite sum of complete vector fields on $\mathbb{C}^n$, its flow can be approximated by elements of the automorphism group of $\mathbb{C}^n$.

Proposition 2.13. *Let X be a holomorphic vector field, and suppose that its flow $\phi(t, z)$ exists on $[0, 1] \times \mathbb{C}^n$. Assume that there exist complete holomorphic vector fields $X_1, \dots, X_m$ such that $X = \sum_{j=1}^m X_j$ and $\phi^k(t, z)$ is the flow of the vector field X_k for $k \in \{1, 2, \dots, m\}$. Then*

$$(\phi^1_{\frac{t}{n}} \circ \phi^2_{\frac{t}{n}} \circ \dots \circ \phi^m_{\frac{t}{n}})^n(z) \to \phi(t, z),$$

uniformly on every compact subset of $[0, 1] \times \mathbb{C}^n$.

Proof. Let $z_0 \in \mathbb{C}^n$ and $V \subset \mathbb{C}^n$ be a relatively compact open neighborhood of z_0. Let $K_t(z) = \phi_t^1 \circ \phi_t^2 \circ \dots \circ \phi_t^m(z)$ and choose $t_0 > 0$ such that $B(z_0, 2Ct_0) \subset V$, where $C := \sup_{t \leq t_0, x \in V} \left\| \frac{\partial}{\partial t} K_t(z) \right\|$. The constant $C > 0$ exists since $\frac{\partial}{\partial t} K_t(z) \to X(z)$ exists as $t \to 0^+$ uniformly on every compact subset on $\mathbb{C}^n$, particularly on V.

We first prove the statement for $t \leq t_0$. We use induction to show that $(K_{\frac{t}{n}})^n(z) := \underbrace{(K_{\frac{t}{n}} \circ K_{\frac{t}{n}} \cdots \circ K_{\frac{t}{n}})}_{n \text{ times}}(z) \in B(z_0, 2Ct_0)$ for every $n \in \mathbb{N}$ and for each $z \in B(z_0, Ct_0)$ and each $t \leq t_0$. Since

$$\|K_t(z) - z\| \leq \int_0^t \left\| \frac{\partial K_\tau(z)}{\partial \tau} \right\| d\tau \leq Ct,$$
$$\forall t \leq t_0 \; \forall z \in B(z_0, Ct_0),$$

we obtain that

$$\|K_t(z) - z_0\| \leq \|K_t(z) - z\| + \|z - z_0\| \leq 2Ct_0, \forall z \in B(z_0, Ct_0),$$

and the statement is proved for $n = 1$.

Let $n > 1$ and $z \in B(z_0, Ct_0)$. We can then express $(K_{\frac{t}{n}})^n$ as a telescopic sum in the following way:

$$(K_{\frac{t}{n}})^n(z) - z = \sum_{j=1}^{n} K_{\frac{t}{n}}((K_{\frac{t}{n}})^{n-j}(z)) - (K_{\frac{t}{n}})^{n-j}(z) \quad (2.5)$$

Since from the induction hypothesis $(K_{\frac{t}{n}})^{n-j}(z) \in V$, for all $z \in B(z_0, Ct_0)$, hence,

$$\|K_{\frac{t}{n}}((K_{\frac{t}{n}})^{n-j}(z)) - (K_{\frac{t}{n}})^{n-j}(z)(z)\|$$
$$\leq \int_0^{\frac{t}{n}} \left\| \frac{\partial}{\partial \tau} K_\tau((K_{\frac{t}{n}})^{n-j}(z))\, d\tau \right\| \cdot \leq C\frac{t}{n}. \quad (2.6)$$

From (2.5), (2.6) it follows that $\|(K_{\frac{t}{n}})^n(z) - z\| < Ct_0$. Hence, using triangle inequality we get that

$$\|(K_{\frac{t}{n}})^{\frac{t}{n}}(z) - z_0\| \leq 2Ct_0.$$

It follows from Lemma 2.9 that $\frac{\partial}{\partial t}\big|_{t=0} K_t(z) = X(z)$. From Result 2.8, we get that $(\phi_t(z) - z - tX(z)) = o(t)$. Also, in view of Lemma 2.9, we obtain that $(K_t(z) - z - tX(z)) = o(t)$. Hence,

$$\phi_t(z) - K_t(z) = (\phi_t(z) - z - tX(z)) - (K_t(z) - z - tX(z))$$
$$= o(t). \quad (2.7)$$

We now express the difference between $\phi_t(z)$ and $(K_{\frac{t}{n}})^n(z)$ again as a telescopic sum, as follow:

$$\phi_t(z) - (K_{\frac{t}{n}})^n(z) = (\phi_{\frac{t}{n}})^n(z) - (K_{\frac{t}{n}})^n(z)$$
$$= \sum_{j=1}^{n} \Big[(\phi_{\frac{t}{n}})^{n-j} \circ (\phi_{\frac{t}{n}}((K_{\frac{t}{n}})^{j-1}(z)))$$
$$- (\phi_{\frac{t}{n}})^{n-j} \circ (K_{\frac{t}{n}})^j(z) \Big] \quad (2.8)$$

Since our vector field is smooth, hence, it is locally Lipschitz continuous. Therefore, using Result 2.6, we get that

$$\|\phi_t(z) - (K_{\frac{t}{n}})^n(z)\| \leq \sum_{j=1}^{n} e^{t\beta \frac{n-j}{n}}$$
$$\times \|\phi_{\frac{t}{n}}((K_{\frac{t}{n}})^{j-1}(z)) - K_{\frac{t}{n}}((K_{\frac{t}{n}})^{j-1}(z))\|, \forall z \in B(z_0, Ct_0).$$

From (2.7), and above equation we get that

$$\|\phi_t(z) - (K_{\frac{t}{n}})^n(z)\| \leq ne^{\beta t} o\left(\frac{t}{n}\right),$$

where β is the Lipschitz constant of the vector field X on the domain V. We have, therefore, established the assertion for $t \leq t_0$.

Let $T > 0$ and $\Gamma := \{\phi_t(z_0) : t \in [0, T]\}$. Since Γ is compact, we can find a relatively compact open neighborhood $W \subset \mathbb{C}^n$ of Γ and $\delta > 0$ such that $(K_{\frac{t}{n}})^n$ converges to ϕ_t uniformly on compact subsets of W for all $t < \delta$. Choose $n_0 \in \mathbb{N}$ such that $\frac{T}{n_0} < \delta$. Therefore, we have the following for $z \in W$

$$\phi_t(z) = (\phi_{\frac{t}{n_0}})^{n_0}(z) = \lim_{l \to \infty} (K_{\frac{t}{ln_0}})^{ln_0}(z), \forall t \leq T.$$

Hence, it follows that the subsequence $(K_{\frac{t}{ln_0}})^{ln_0}$ uniformly converges to $\phi_t(z)$ on W as $l \to \infty$.

Let $N \in \mathbb{N}$, then there exist $p, q \in \mathbb{N}$ such that $p < n_0$ and $N = n_0 q + p$. Clearly, $\frac{T}{n_0 q + p} < \delta$ and $\frac{Tn_0 q}{n_0 q + p} < T$ and $(K_{\frac{T}{nq+p}})^p$ converges to id as $n \to \infty$.

$$(K_{\frac{T}{N}})^N(z) = (K_{\frac{T}{n_0 q + p}})^{n_0 q}((K_{\frac{T}{n_0 q + p}})^p(z))$$
$$= (K_{\frac{Tn_0 q}{n_0 q + p}\frac{1}{n_0 q}})^{n_0 q}((K_{\frac{T}{n_0 q + p}})^p(z)) \to \phi_T(z)$$

uniformly on W. □

The subsequent propositions enables us to approximate the flow of a time-dependent vector field using the flow of another time-independent vector field.

Proposition 2.14. *Let V_t be a time-dependent holomorphic vector field on $\mathbb{C}^n$ and let $\phi(t; s, z)$ be the solution of the following system of ODE:*

$$\frac{d\phi(t; s, z)}{dt} = V_t(\phi(t; s, z))$$
$$\phi(s; s, z_0) = z_0. \quad (2.9)$$

Let $m \in \mathbb{N}$ and consider the vector field $V_m(t, z)$ as follow:

$$V_m(t, z) = \begin{cases} V(0, z) & \text{if } 0 \leq t < \frac{1}{m} \\ V(\frac{1}{m}, z) & \text{if } \frac{1}{m} \leq t < \frac{2}{m} \\ \vdots & \\ V(\frac{m-1}{m}, z) & \text{if } \frac{m-1}{m} \leq t \leq 1 \end{cases}$$

and let $y_m(t,z)$ be the solution of the following ODE

$$\begin{cases} \frac{dy_m(t,z)}{dt} = V_m(t, y_m(t,z)) \\ y_m(0,z) = z. \end{cases}$$

Then $\{y_m\}_m$ converges uniformly on compact sets of $[0,1] \times \mathbb{C}^n$ (where the flow of the vector field V_t exists) to the flows of the time dependent vector field V_t.

Proof. We will prove the case when $s = 0$. We apply Lemma 2.12 to the vector fields V and V_m. Since the vector field V_m is not continuous, we will have to restrict ourselves to intervals of the form $\left[\frac{k}{m}, \frac{k+1}{m}\right]$ with $0 \le k < m$. Let $K \subset \mathbb{C}^n$ be any compact set, and let $\varepsilon > 0$. Let

$$L = \sup_{(t,z)\neq(t,w)\in[0,1]\times K} \frac{\|V(t,z) - V(t,w)\|}{\|z-w\|},$$

and

$$M_m = \sup_{(t,z)\in[0,1]\times K} \|V(t,z) - V_m(t,z)\|$$

Let $\alpha = \max\left(e^L, \frac{e^L-1}{L}\right)$. Clearly, $\alpha \ge 1$. Since V is continuous, it is absolutely continuous in $[0,1] \times K$. Therefore, there exists $m_0 \in \mathbb{N}$ such that for every $m \ge m_0$, $M_m < \frac{\varepsilon}{2\alpha}$.

Claim 2.15. *If $\phi(t,z)$ is the solution (2.9) with $\phi(0,z) = z$. Then, for each $m > m_0$ and $0 \le k < m$, we have that $\|\phi(t,z) - y_m(t,z)\| < \varepsilon$ in $\left[\frac{k}{m}, \frac{k+1}{m}\right] \times K$.*

We prove the claim by mathematical induction on k.

Let $k = 0$. Here V_m is continuous in $\left[0, \frac{1}{m}\right]$ and V is particularly continuous and locally Lipschitz on z uniformly with respect to t (every holomorphic map is locally Lipschitz). Hence, invoking Lemma 2.12 we get that for every $(t,z) \in \left[0, \frac{1}{m}\right] \times K$,

$$\begin{aligned} \|\phi(t,z) - y_m(t,z)\| &\le \frac{M_m}{L}\left(e^{Lt} - 1\right) \\ &< \frac{e^L - 1}{L}\frac{\varepsilon}{2\alpha} \\ &\le \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

Now suppose the result is true for k and let us prove it for $k+1$. For every $(t,z) \in \left[\frac{k+1}{m}, \frac{k+2}{m}\right] \times K$ we have $t = \tau + \frac{k+1}{m}$ for some $\frac{1}{m} > \tau \ge 0$. From the property of the flow map we have the following:

$$\begin{aligned} &\|\phi(t,z) - y_m(t,z)\| \\ &\quad = \left\|\phi\left(\tau, \left(\phi\left(\frac{k+1}{m}, z\right)\right)\right) - y_m\left(\tau, y_m\left(\frac{k+1}{m}, z\right)\right)\right\|. \end{aligned}$$

In view of Lemma 2.12, we obtain that

$$\begin{aligned} &\|\phi(t,z) - y_m(t,z)\| \\ &\le \left\|\phi\left(\frac{k+1}{m}, z\right) - y_m\left(\frac{k+1}{m}, z\right)\right\| e^{L\tau} + \frac{M_m}{L}(e^{L\tau} - 1) \\ &\le \left\|\varphi\left(\frac{k+1}{m}, z\right) - y_m\left(\frac{k+1}{m}, z\right)\right\| e^{L} + \frac{M_m}{L}(e^{L} - 1) \\ &< \frac{\varepsilon}{2\alpha}\frac{e^L-1}{L}e^L + \frac{e^L-1}{L}\frac{\varepsilon}{2\alpha}. \end{aligned}$$

From the choice of $\alpha > 0$ we have

$$\|\phi(t,z) - y_m(t,z)\| \le \varepsilon.$$

Therefore, for any $0 \le k < m$ we have

$$\|\phi(t,z) - y_m(t,z)\|_{\left[\frac{k}{m}, \frac{k+1}{m}\right]\times K} < \varepsilon,$$

for all $m \ge m_0$. Consequently, we have

$$\|\phi(t,z) - y_m(t,z)\|_{[0,1]\times K} < \epsilon.$$

That is, $(y_m)_m$ converges uniformly on compact subsets of $[0,1] \times \mathbb{C}^n$ to ϕ. □

The following proposition will be used to prove volume-preserving approximation. Recall that for vector field $F = (F_1, F_2, \ldots, F_n)$, $\mathrm{div}F := \frac{\partial F_1}{\partial z_1} + \frac{\partial F_2}{\partial z_2} + \cdots + \frac{\partial F_n}{\partial z_n}$.

Proposition 2.16. *Let $Y : \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic vector field with* div $Y = 0$*. Then, there exists a sequence of polynomial vector fields $Z_m \colon \mathbb{C}^n \to \mathbb{C}^n$ approximating Y uniformly on compact sets such that* $\mathrm{div}Z_m = 0$.

Proof. Fix $K \subset \mathbb{C}^n$ compact and $\varepsilon > 0$. Let $Y(z) = (y_1(z), \ldots, y_n(z))$ and assign the holomorphic $(n-1,0)$-form as follow

$$\omega(z) = \sum_{k=1}^{n} (-1)^{k-1} Y_k(z) dz_1 \wedge \ldots \wedge \widehat{dz_k} \wedge \ldots \wedge dz_n.$$

Now exterior derivative of ω is

$$d\omega = \sum_{k=1}^{n} (-1)^{k-1} \frac{\partial Y_k}{\partial z_k} dz_k \wedge dz_1 \wedge \ldots \wedge \widehat{dz_k} \wedge \ldots \wedge dz_n$$

$$= \sum_{k=1}^{n} \frac{\partial Y_k}{\partial z_k} dz_1 \wedge \ldots \wedge dz_n$$

$$= \operatorname{div} Y dz_1 \wedge \ldots \wedge dz_n.$$

Hence, from assumption we have

$$d\omega = 0.$$

Since $\mathbb{C}^n$ is simply connected, there exists a holomorphic $(n-2,0)$-form τ such that $\omega = d\tau$. As τ is holomorphic, hence, we can find a sequence of $(n-2,0)$-forms with polynomial coefficients approximating τ uniformly on compact subsets of $\mathbb{C}^n$.

If

$$\tau = \sum_{1 \le i < j \le n} \tau_{ij} dz_1 \wedge \ldots \wedge \widehat{dz_i} \wedge \ldots \wedge \widehat{dz_j} \wedge \ldots \wedge dz_n$$

then given any $\varepsilon > 0$ and any compact subset $K \subset \mathbb{C}^n$ it follows that there exists

$$\sigma = \sum_{1 \le i < j \le n} \sigma_{ij} dz_1 \wedge \ldots \wedge \widehat{dz_i} \wedge \ldots \wedge \widehat{dz_j} \wedge \ldots \wedge dz_n,$$

such that $| \tau_{ij}(z) - \sigma_{ij}(z) | < \varepsilon$. Here computing the exterior derivative of both τ and σ we get that

$$\omega = d\tau = \sum_{1 \le i < j \le n} (-1)^{i-1} \frac{\partial \tau_{ij}}{\partial z_i} dz_1 \wedge \ldots \wedge \widehat{dz_j} \wedge \ldots \wedge dz_n$$

$$+ \sum_{1 \le i < j \le n} (-1)^{j-2} \frac{\partial \tau_{ij}}{\partial z_j} dz_1 \wedge \ldots \wedge \widehat{dz_i} \wedge \ldots \wedge dz_n$$

and

$$d\sigma = \sum_{1 \le i < j \le n} (-1)^{i-1} \frac{\partial \sigma_{ij}}{\partial z_i} dz_1 \wedge \ldots \wedge \widehat{dz_j} \wedge \ldots \wedge dz_n$$

$$+ \sum_{1 \le i < j \le n} (-1)^{j-2} \frac{\partial \sigma_{ij}}{\partial z_j} dz_1 \wedge \ldots \wedge \widehat{dz_i} \wedge \ldots \wedge dz_n.$$

Finally, rewrite $d\sigma$ as

$$d\sigma = \sum_{k=1}^{n} (-1)^{k-1} Z_k(z) dz_1 \wedge \ldots \wedge \widehat{dz_k} \wedge \ldots \wedge dz_n.$$

Let $Z(z) = (Z_1(z), \ldots, Z_n(z))$ be a holomorphic polynomial vector field. Here, the vector field Z is constructed from the $(n-1,0)$-form $d\sigma$ in similar way the $(n-1,0)$-form ω constructed from the vector field Y. Therefore, Z approximate Y uniformly on K.

Also, we have

$$0 = d^2\sigma = \sum_{k=1}^{n} \frac{\partial Z_k}{\partial z_k} dz_1 \wedge \ldots \wedge dz_n = \operatorname{div} Z dz_1 \wedge \ldots \wedge dz_n.$$

Consequently, $\operatorname{div} Z = 0$. This completes the proof. □

The next proposition holds crucial significance in establishing the proof of both Theorem 2.4 and Theorem 3.3, enabling us to break down every algebraic vector field into a finite sum of complete vector fields.

Proposition 2.17 ([6, Lemma 4.9.9]). *For each $k \in \mathbb{N}$ there exists finitely many $\mathbb{C}$-linear functionals $\lambda_1, \lambda_2, \ldots, \lambda_r$ on $\mathbb{C}^n$ and vectors $v_1, v_2, \ldots, v_r \in \mathbb{C}^n$, with $\lambda_j(v_j) = 0$ and $\|v_j\| = 1$ for all $j \in \{1, 2, \ldots, r\}$, such that every holomorphic polynomial map $V : \mathbb{C}^n \to \mathbb{C}^n$ that is homogeneous of degree k is of the form*

$$V(z) = \sum_{j=1}^{r} c_j (\lambda_j(z))^k v_j + d_j (\lambda_j(z))^{k-1} \langle z, v_j \rangle v_j, \quad (2.10)$$

for some $c_j, d_j \in \mathbb{C}$. If $\operatorname{div}(wV) = 0$, d_j can be taken 0 for all $j \in \{1, 2, \ldots, r\}$.

Proof. Let $P \in \mathbb{C}[z_1, \ldots, z_n]$ is a homogeneous polynomial of degree k. Invoking Lemma 2.10, we get that there exist linear forms $\lambda_i(z) = \sum_{j=1}^{n} a_{ij} z_j$ on $\mathbb{C}^n$ $(i = 1, \ldots, m = m(k,n))$ such that

$$P(z) = \sum_{i=1}^{m} c_i \lambda_i(z)^k, \quad c_i \in \mathbb{C}. \quad (2.11)$$

The forms λ_i may be chosen from any nonempty open set $U \subset (\mathbb{C}^n)^*$.

Let $e_1, e_2, \ldots, e_n$ are the standard ordered basis of $\mathbb{C}^n$. By a linear change of coordinates, we may assume that $e_1^*(z) := z_1 \in U$, where $e_1^* : \mathbb{C}^n \to \mathbb{C}$ defined by $e_1^*(e_j) = 0$ for all $j \in \{1, 2, \ldots, n\}$ and $e_1^*(e_1) = 1$. Choose $\lambda_1, \lambda_2, \ldots, \lambda_m \in U$ satisfying (2.11) for homogeneous polynomials of degree k and $k-1$. After a linear change of coordinates we can assume that $\lambda_i(e_n) = a_{in} = 1$ for all $i = \{1, 2, \ldots, m\}$.

Now $\operatorname{div} V(z)$ is a homogeneous polynomial of degree $k-1$. Therefore, from (2.11), we conclude that there exist $d_j \in \mathbb{C}$ such that $\operatorname{div} V(z) = \sum_{j=1}^{m} d_j (\lambda_j(z))^{k-1}$. Choose $v_j := (v_{j1}, v_{j2}, \ldots, v_{jn}) \in \operatorname{Ker} \lambda_j$ with $\|v_j\| = 1$ and set

$$\widetilde{V}_j(z) = d_j (\lambda_j(z))^{k-1} \langle z, v_j \rangle v_j$$

for $j = 1, 2, \ldots, m$. Next, we compute the divergence of $\widetilde{V}_j$:

$$\mathrm{div}\widetilde{V}_j(z) = \sum_{\alpha=1}^{n} v_{j\alpha}\big(d_j(k-1)(\lambda_j(z))^{k-2}a_{j\alpha}\langle z, v_j\rangle + d_j(\lambda_j(z))^{k-1}\overline{v_{j\alpha}}\big).$$

Since $v_j \in \mathrm{Ker}\lambda$ and $\|v_j\|^2 = 1$, hence, we have

$$\mathrm{div}\widetilde{V}_j(z) = d_j(\lambda_j(z))^{k-1} \qquad \forall j \in \{1, 2, \ldots m\}.$$

Since $\mathrm{div}V = \sum_{j=1}^{m}\mathrm{div}\widetilde{V}_j(z)$, therefore, we have that $\mathrm{div}X(z) = 0$, where $X(z) = V - \sum_{j=1}^{m}\widetilde{V}_j(z)$. Consequently, we have $V = X + \sum_{j=1}^{m}\widetilde{V}_j(z)$ and the form of $\widetilde{V}_j(z)$ as second summand of the (2.10). Therefore, it is enough to show that for every divergence zero homogeneous polynomial vector field X on $\mathbb{C}^n$ of degree k there exists $\lambda_i \in U$ and $v_i \in \mathrm{Ker}\lambda_i$ and $r \in \mathbb{N}$ such that

$$X(z) = \sum_{i=1}^{r} c_i(\lambda(z))^k v_i, \; c_i \in \mathbb{C}. \tag{2.12}$$

We apply (2.11) to the component X_l of X for $l \in \{1, \ldots, n-1\}$ to get $X_l(z) = \sum_{i=1}^{m} c_{il}(\lambda_i(z))^k$. Since from the construction of λ_i we have $\lambda_i(e_l - a_{il}e_n) = 0$, hence the flow of each vector field defined by

$$V_{il}(z) := c_{il}\lambda_i(z)^k(e_l - a_{il}e_n)$$

is a shear map. Also $\mathrm{div}V_{il} = 0$ for all $i \in \{1, 2, \ldots, m\}$ and $l \in \{1, 2, \ldots, (n-1)\}$. Let $W = \sum_{i=1}^{m}\sum_{l=1}^{n-1} V_{i,l} = (W_1, \ldots, W_n)$. Then, $W_l = X_l$ for $l = 1, \ldots, n-1$, and hence $X = W + (X_n - W_n)e_n$.

Since $\mathrm{div}X = 0 = \mathrm{div}W$, we get $\frac{\partial(X_n - W_n)}{\partial z_n} = 0$. Therefore, $X_n - W_n$ is independent of z_n, and hence $(X_n - W_n)e_n$ is a vector field whose flow is a shear map of $\mathbb{C}^n$.

By choosing another linear form $\lambda_j'(z) \in U$ that depends on $(z_1, z_2, \ldots, z_{n-1})$ we write $X_n - W_n = \sum_j c_j'(\lambda_j'(z_1, z_2 \ldots, z_{n-1}))^k$. Therefore,

$$X = W + \sum_j c_j'\lambda_j'(z_1, z_2 \ldots, z_{n-1})e_n.$$

By construction, W is the first summand of the (2.10). Hence, we get the required form of X. This completes the proof. □

Remark 2.18. The flow of the vector field $c_j(\lambda_j(z))^k v_j$ is of the form $\Phi_t(z)$ defined as (2.1) with the function $f\colon \mathbb{C} \to \mathbb{C}$ defined by $f(z) = c_j z^k$. Similarly, the flow of the vector field $d_j(\lambda_j(z))^{k-1}\langle z, v_j\rangle v_j$ is of the form $\Psi_t(z)$ as given in (2.2) with $f\colon \mathbb{C} \to \mathbb{C}$ defined by $f(z) = d_j z^{k-1}$.

Next, we point out the main steps in the proof of Theorem 2.4.

Step 1. *First, we will establish a homotopy between $f \in Aut(\mathbb{C}^n)$ and the identity through a map $\phi(t, z)$ such that $\phi(0, z) = z$, $\phi(1, z) = f(z)$, and ϕ is the solution to some time-dependent vector field X (which also will depend on f). Thus, f will be the time 1 map of the flow map of X.*

At this stage, the approach is to approximate the vector field X in intervals of the form $[\frac{k}{m}, \frac{k+1}{m}]$ by time-independent vector fields $V_{k,m}$. From Proposition 2.14, it will imply that a solution of $V_{k,m}$ will approximate the solution of X, denoted by ϕ, in the interval $[\frac{k}{m}, \frac{k+1}{m}]$. Consequently, concatenating the solutions of $V_{k,m}$ will yield an approximation to ϕ.

Step 2. *Next, we will approximate each vector field $V_{k,m}$ using a vector field whose solutions exhibit overshearing behavior. To accomplish this we will do the following:*

(1) We will initially approximate $V_{k,m}$ by employing a polynomial vector field.

(2) Subsequently, invoking Proposition 2.17 we will decompose the polynomial vector field into a combination of complete vector fields Z_j, ensuring that their time-t maps manifest as overshears.

Step 3. *The final step involves composing the flows of the Z_j's at time $t > 0$. Invoking Proposition 2.13, we get an approximation (comprising a composition of overshears) to the solution of $V_{k,m}$, providing us with an approximation to ϕ. Setting $t = 1$, we obtain an approximation to f.*

The following lemma is also required for the proof.

Lemma 2.19. *Let $f \in Aut(C^n)$ with $f(0) = 0$ and $Df(0) = Id$. Then $\phi\colon \mathbb{R} \times \mathbb{C}^n \to \mathbb{C}^n$ defined by*

$$\phi(t, z) = \begin{cases} \frac{f(tz)}{t} & \text{if } t \neq 0, \\ z, & \text{if } t = 0, \end{cases}$$

is a 1-parameter group of automorphisms of $\mathbb{C}^n$, holomorphic in both t and z variables. Moreover, ϕ satisfies the following ODE

$$\begin{cases} \frac{d\phi(t,z)}{dt} = X(t, (\phi(t, z))), \\ \phi(0, z) = z, \end{cases} \tag{2.13}$$

where

$$X(t, z) = \frac{d\phi}{dt}(t, \phi_t^{-1}(z)),$$

thus, in particular, X is complete. If, in addition, $\det Df(z) \equiv c$ for some constant $c \in \mathbb{R}$, then $divX = 0$.

Proof. It is clear from the definition that the map $\phi_t : \mathbb{C}^n \to \mathbb{C}^n$ defined by $\phi_t(z) := \phi(t,z) \in \mathrm{Aut}(\mathbb{C}^n)$ for all $t \in [0,1]$. Hence, ϕ is a one-parameter family of automorphisms. Since $f \in \mathrm{Aut}(\mathbb{C}^n)$, hence, the Taylor series expansion of the mapping f at the origin converges uniformly on every compact subset of $\mathbb{C}^n$. Since $f(0) = 0$ and $Df(0) = Id$, hence, we get that

$$f(tz) = tz + \sum_{k=2}^{\infty} \frac{1}{k!} D^k f(0) \underbrace{(tz, tz, \ldots, tz)}_{k-\text{times}}, \tag{2.14}$$

where $D^k f(0) \colon \underbrace{\mathbb{C}^n \times \mathbb{C}^n \cdots \times \mathbb{C}^n}_{k-\text{times}} \to \mathbb{C}^n$ is k-linear map over $\mathbb{C}$.

$$\tag{2.15}$$

Thus, we have

$$\frac{f(tz)}{t} = z + \sum_{k=2}^{\infty} t^{k-1} \frac{1}{k!} D^k f(0) \underbrace{(z, z, \ldots, z)}_{k-\text{times}}. \tag{2.16}$$

Clearly, $\sum_{k=2}^{\infty} t^{k-1} \frac{1}{k!} D^k f(0) \underbrace{(z, z, \ldots, z)}_{k-\text{times}} \to 0$ uniformly over every compact subset to 0 as $t \to \infty$. Hence, the map $\phi_t(\cdot)$ is holomorphic for every $t \in [0,1]$. It follows from the definition of X that ϕ satisfies the (2.13).

If $\det Df(z)$ is constant, then it follows that $D_z\phi_t(z) = Df(tz) =$ constant. Hence, from Result 2.7, it follows that $\mathrm{div} X = 0$. □

Proof of Theorem 2.4. Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be an automorphism of $\mathbb{C}^n$. Fix any compact $K \subset \mathbb{C}^n$ and $\epsilon > 0$. Without loss of generality we can suppose that $f(0) = 0$ and $Df(0) = \mathrm{Id}$. Otherwise, we will consider the map $g(z) = Df(0)^{-1}(f(z) - f(0))$.

From [2, Proposition 3.1] it follows that any invertible linear map (and translation) can be written as a composition of overshears. Therefore, it is enough to prove that g can be approximated by $\mathrm{Aut}(\mathbb{C}^n)$. Thus, we can assume that $f(0) = 0$ and $Df(0) = \mathrm{Id}$. Then, by Lemma 2.19, $\phi(t,z) = \frac{f(tz)}{t}$ if $t \neq 0$ and $\phi(0,z) = z$ is a 1-parameter group of automorphisms of $\mathbb{C}^n$. Moreover, $\phi(t,z)$ can be viewed as the flow of the time-dependent vector field X defined as Lemma 2.19. Next, we define the following vector field

$$X_m(t,z) = \begin{cases} X(0,z) & \text{if } 0 \le t < \frac{1}{m} \\ X(\frac{1}{m}, z) & \text{if } \frac{1}{m} \le t < \frac{2}{m} \\ \vdots & \\ X(\frac{m-1}{m}, z) & \text{if } \frac{m-1}{m} \le t \le 1. \end{cases}$$

Suppose that $x_m(t,z)$ be the solution of the following system:

$$\begin{cases} \frac{dx_m(t,z)}{dt} = X_m(t, x_m(t,z)) \\ x_m(0,z) = z. \end{cases}$$

Invoking Proposition 2.14, we get that x_m approximates ϕ uniformly on K. Thus, there exists $m_0 \in \mathbb{N}$ such that for all $m \ge m_0$ we have

$$\|\phi(t,z) - x_m(t,z)\|_{[0,1]\times K} < \frac{\varepsilon}{3}. \tag{2.17}$$

Since $X_m(t,z)$ is a time-independent in the intervals of the form $[k/m, (k+1)/m]$ and holomorphic in z variables for $z \in \mathbb{C}^n$, hence, we can approximate X_m uniformly on $[k/m, (k+1)/m] \times K$ by a polynomial vector field $Z^{(k)}$ whose flow $\varphi_{(k)}$ satisfies

$$\|x_m(t,z) - \varphi_{(k)}(t,z)\|_{[k/m,(k+1)/m]\times K} < \frac{\varepsilon}{3}. \tag{2.18}$$

Now Proposition 2.17, allows us to write the polynomial vector field $Z^{(k)}$ as finite sum complete polynomial vector field as follow:

$$Z^{(k)} = \sum_{j=1}^{m} Z_j^{(k)},$$

where $Z_j^{(k)}$ are complete vector field for $j = 1, 2, \ldots m$.

Suppose that $\eta^j_{(k)}(x) := (\psi^1_{\frac{t}{j}} \circ \psi^2_{\frac{t}{j}} \circ \cdots \circ \psi^m_{\frac{t}{j}})^j(x)$, where $\psi^j_t(z)$ is the flow of the vector field $Z_j^{(k)}$ for $j = 1, 2, \ldots, m$. Then, from Lemma 2.19, it follows that there exists $j_k \in \mathbb{N}$ such that for all $j \ge j_k$ and for all $(t,z) \in \left[\frac{k}{m}, \frac{k+1}{m}\right] \times K$ it follows that,

$$\|\varphi_{(k)}(t,z) - \eta^j_{(k)}(t,z)\| < \frac{\varepsilon}{3}. \tag{2.19}$$

Therefore, for all $j \ge j_k$ we have the following: For all $(t,z) \in [\frac{k}{m}, \frac{k+1}{m}] \times K$,

$$\begin{aligned} &\|\phi(t,z) - \eta^j_{(k)}(t,z)\| \\ &\quad \le \|\phi(t,z) - x_m(t,z)\| + \|x_m(t,z) - \varphi_{(k)}(t,z)\| \\ &\qquad + \|\varphi_{(k)}(t,z) - \eta^j_{(k)}(t,z)\| \\ &\quad < \varepsilon. \end{aligned} \tag{2.20}$$

This implies that for every $t > 0$, $\eta^j_{(k)}(t,z) \to \phi(t,z)$ uniformly over K as $j \to \infty$. (Here k depends on t).

It follows from Remark 2.18 that the $\eta^j(t,z)$ is a composition of overshears.

Therefore, taking $t = 1$, we get that $\phi(1,z) = f(z)$ can be approximated by the composition of overshears.

If we assume that $\det(Df(z)) \equiv 1$ then by Lemma 2.19, we get that $\mathrm{div}X(z) = 0$. Then, from Proposition 2.16, we get that we can choose the polynomial vector field Z^k_j such that $\mathrm{div}Z^k_j(z) = 0$. In view of Proposition 2.17, we will get that map $\eta^j(t,z)$ is composition of shear. $\square$

3. Approximation of biholomorphisms by automorphisms of $\mathbb{C}^n$

In this section, we discuss about the approximation of the biholomorphic maps from domains in $\mathbb{C}^n$ by finite composition of shears and overshears. This can be thought of a generalization of the corresponding question in Section 2, where the domain is the whole of $\mathbb{C}^n$. The question was also considered by Andersén and Lempert [2]. In order to go further into our discussion we need the notion of the Runge domain, which is a fundamental concept in the function theory of several complex variables.

Definition 3.1. *A domain $\Omega \subseteq \mathbb{C}^n$ is said to be a Runge domain if every holomorphic function $f : \Omega \to \mathbb{C}$ can be approximated by holomorphic polynomials in $z_1, z_2, \ldots, z_n$ uniformly on every compact subset of Ω.*

In one variable Runge domains are the domains that are simply connected. In higher dimensions, there is no such characterization of Runge domains. There are some known classes of domains that are Runge; for instance, convex domains, polynomial polyhedra etc. From Hömander's theorem, it follows that the domains of the form $\{z \in \mathbb{C}^n | \varphi(z) < \alpha\}$, where $\alpha > 0$ and $\varphi\colon \mathbb{C}^n \to \mathbb{R}$ is a plurisubharmonic (i.e. subharmonic in each complex line) function on $\mathbb{C}^n$, are Runge. In [5], Al kasimi proved that any starshaped domain is Runge. Recently, Hamada [9, Theorem-3.1] proved that for a given matrix A with $\inf_{\|z\|=1}\{\langle Az, z\rangle\} > 0$, the domains $\Omega \subset \mathbb{C}^n$ with $e^{-tA}\Omega \subseteq \Omega$, for all $t \geq 0$, are Runge. These domains are called *spirallike domains* with respect to A. It is proved in [4] that a spiralllike domain with respect to an asymptotically stable vector field is Runge. The reader is referred to [4] for details.

Let us now proceed with our discussion of the approximation of biholomorphism for the starshaped domain. The following theorem was proved by Andersén and Lempert [2, Theorem 2.1].

Theorem 3.2 (Andersén, Lempert 1992, [2]). *Let Ω be a starshaped domain $\mathbb{C}^n$ and $\Phi : \Omega \to \mathbb{C}^n$ be a biholomorphic map such that $\Phi(\Omega)$ is Runge. Then Φ can be approximated uniformly on compact subsets by compositions of shears and overshears. If $\det D\Phi \equiv 1$, then Φ can be approximated uniformly on compact subsets by the composition of shears.*

Note that the Runge property of the domain and the range of Φ in Theorem 3.2 is crucial. The crux of the above theorem is taken by Forstnerič and Rosay [7] and formulated the following general result that found several applications.

Theorem 3.3 (Forstnerič Rosay 1993, [7]). *Let Ω be an open set in $\mathbb{C}^n (n \geq 2)$. For every $t \in [0,1]$, let Φ_t be a biholomorphic map from Ω into $\mathbb{C}^n$ of class C^2 in $(t,z) \in [0,1] \times \Omega$. Assume that each domain $\Omega_t = \Phi_t(\Omega)$ is Runge in $\mathbb{C}^n$.*

If Φ_0 can be approximated on Ω by holomorphic automorphisms of $\mathbb{C}^n$, then for every $t \in [0,1]$ the map Φ_t can be approximated on Ω by holomorphic automorphism of $\mathbb{C}^n$. If in addition Ω is a domain of holomorphy satisfying $H^{n-1}(\Omega;\mathbb{C}) = 0$ and if every Φ_t is volume preserving (i.e., its Jacobian determinant equals one), and if Φ_0 can be approximated on Ω by volume preserving automorphisms of $\mathbb{C}^n$, then every Φ_t can be approximated on Ω by volume preserving automorphisms of $\mathbb{C}^n$.

Conversely, if Ω is a pseudoconvex Runge domain in $\mathbb{C}^n$ and $\Phi_1 \colon \Omega \to \mathbb{C}^n$ is a biholomorphic map that can be approximated on Ω by automorphisms of $\mathbb{C}^n$, then for every compact set $K \subset \Omega$ there is an open set D, such that $K \subset D \subset \Omega$, and a family of biholomorphic maps $\Phi_t \colon D \to \mathbb{C}^n$, of class C^∞ in $(t,z) \in [0,1] \times D$, such that Φ_0 is the identity map, Φ_1 is the given map, and every Φ_t can be approximated by automorphisms of $\mathbb{C}^n$, and $\Phi_t(D)$ is Runge in $\mathbb{C}^n$ for each $t \in [0,1]$.

The class of domains in $\mathbb{C}^n$ for which the biholomorphic maps can be approximated by automorphisms of $\mathbb{C}^n$ (equivalently by finite compositions of shears and overshears) are important. Applying Theorem 3.2, Arosio-Bracci-Wold [3] found a solution to the following open problem for the case of the starshaped domain.

Open question: *Given a Herglotz vector field $G(z,t)$ of order $d \in [1,+\infty]$ on D, does there exist a univalent solution ($f_t : D \to \mathbb{C}^n$) of the following Loewner PDE:*

$$\frac{\partial f_t(z)}{\partial t} = -df_t(z)(G(z,t)). \tag{3.1}$$

Equivalently, does the Loewner range R of the Herglotz vector field $G(z,t)$ embed as a domain in $\mathbb{C}^n$?

For the notion of Herglotz vector field and Loewner theory see [3,8]. It follows from the proof of [3, Theorem 3.4] that, if $\Omega \subset \mathbb{C}^n$ is a Runge domain such that every biholomorphism of Ω with Runge image can be approximated by holomorphic automorphisms of $\mathbb{C}^n$, then the associated Loewnwer range of a given Herglotz vector field G defined on the domain Ω can be embedded as a domain in $\mathbb{C}^n$.

While the Theorem 3.3 provides a sufficient condition for approximating a biholomorphism by automorphisms of $\mathrm{Aut}(\mathbb{C}^n)$, finding an isotopy map that satisfies this condition for a given domain Ω and biholomorphism is by no means a straightforward task. Identifying such an isotopy Φ_t in practice can be quite challenging. We now point out few generalizations of Theorem 3.2 where the explicit isotopy is constructed for certain classes of domains. For a square matrix A let $\sigma(A)$ denote the set of all eigenvalues of the matrix A and $k_+(A) := \max\{\mathrm{Re}\lambda | \lambda \in \sigma(A)\}$, $k_-(A) := \min\{\mathrm{Re}\lambda | \lambda \in \sigma(A)\}$.

The following result, due to Hamada [9], reveals that, on a spirallike domain with respect to a certain linear vector field, the Theorem 3.3 can be applied.

Theorem 3.4 ([9, Theorem 4.2]). *Let Ω be a domain containing the origin that satisfies the following conditions: $e^{-tA}w \in \Omega$ for all $w \in \Omega$, where $A \in M_n(\mathbb{C})$ such that $\inf_{\|z\|=1} \mathrm{Re}\langle Az, z\rangle > 0$. If $k_+(A) < 2\inf_{\|z\|=1} \mathrm{Re}\langle Az, z\rangle$, where $\sigma(A)$ is the spectrum of A, then any biholomorphism $\phi : \Omega \to \mathbb{C}^n$ with $\phi(\Omega)$ is Runge, can be approximated uniformly on compacts of Ω by automorphisms of $\mathbb{C}^n$.*

The results presented in [4, Theorem 1.9] generalize Theorem 3.4 by extending the scope of domains in which every biholomorphism with a Runge image can be approximated by automorphisms of $\mathbb{C}^n$. We mention one result here from [4].

Theorem 3.5. *Let $A \in GL(n,\mathbb{C})$ with $2k_+(A) < k_-(A)$ and $\Omega \subset \mathbb{C}^n$ be a domain containing the origin and spirallike with respect to $P^{-1}AP$ for some $P \in GL(n,\mathbb{C})$. Assume that $\Phi\colon \Omega \to \Phi(\Omega)$ is biholomorphism with $\Phi(\Omega)$ Runge. Then:*

(i) The map Φ can be approximated by elements of $Aut(\mathbb{C}^n)$ uniformly on every compact subset of Ω.

(ii) If Ω is pseudoconvex and $J\Phi \equiv 1$ then Φ can be approximated by elements of $Aut_1(\mathbb{C}^n)$ uniformly on every compact subsets of Ω.

(iii) Let $n = 2m$ and $(z_1, z_2, \ldots, z_m, w_1, w_2, \ldots, w_m)$ be a coordinate of $\mathbb{C}^n$. Assume that $\Omega \subseteq \mathbb{C}^n$ is pseudoconvex domain and $\omega = \sum_{j=1}^m dz_i \wedge dw_i$ is a symplectic form on $\mathbb{C}^n$. If $A \in Sp(m,\mathbb{C}) := \{A \in Gl(n,\mathbb{C}) : A^tJA = A\}$, where $J = \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix}$ and $\Phi^*(\omega) = \omega$ then Φ can be approximated by $Aut_{sp}(\mathbb{C}^n) := \{\Psi \in Aut(\mathbb{C}^n) : \Psi^*(\omega) = \omega\}$, where ω is the standard symplectic form on $\mathbb{C}^n$.

The following example from [4, Example 6.13] proves that Theorem 3.5 enlarges the class of domains where every biholomorphic with Runge image can be approximated by elements of $\mathrm{Aut}(\mathbb{C}^n)$.

Example 3.6. *Let $A = \lambda I_n + \mathcal{N}$, where $\mathcal{N}$ is a $n \times n$ matrix with ones in the first diagonal above the main diagonal and zeros elsewhere and $\lambda = \lambda_1 + i\mu_1$, with $\lambda_1 < 0$. Then $\Omega := \{(z_1, z_2, \ldots, z_n) : \left|z_{n-1} - \frac{z_n}{\lambda_1} \ln|z_n|\right| < 1\}$ is a spirallike domain with respect to A.*

Similar results about non-linear vector fields are achieved in [4, Theorem 1.10].

Another application of the approximation of biholomorphic maps is the existence of the Fatou–Bieberbach domain.

Definition 3.7 (Fatou–Bieberbach Domain). *A proper sub-domain Ω of $\mathbb{C}^n$ that is biholomorphic to $\mathbb{C}^n$ is called a Fatou-Bieberbach domain. A biholomorphic map $F\colon \mathbb{C}^n \to \Omega$ onto such Ω (and its inverse map) is called a Fatou-Bieberbach map.*

In view of the Riemann mapping theorem, this phenomenon happens only when $n > 1$. Let $F \in \mathrm{Aut}(\mathbb{C}^n)$ and let F^k denote its k-th iterate of the map F define by $F^k := \underbrace{F \circ F \circ \cdots \circ F}_{k\text{-times}}$.

A point $p \in \mathbb{C}^n$ for which $F(p) = p$ is called a fixed point of F. A fixed point p is said to be attracting if all eigenvalues

λ_j of the derivative $F'(p)$ satisfy $|\lambda_j| < 1$, and is said to be repelling if all eigenvalues satisfy $|\lambda_j| > 1$. In the attracting case, the set

$$\Omega_{F,p} := \{z \in \mathbb{C}^n : \lim_{k\to\infty} F^k(z) = p\},$$

is known as the basin of attraction of the point p.

Theorem 3.8 ([6, Theorem 4.3.2]). *If $n > 1$ and $p \in \mathbb{C}^n$ is an attracting fixed point of a holomorphic automorphism $F \in Aut(\mathbb{C}^n)$, then the attracting basin Ω of $F \in Aut(\mathbb{C}^n)$(defined as above) is Runge. Moreover, there exists a biholomorphic map ψ from Ω onto $\mathbb{C}^n$. If the Jacobian JF is constant, then ψ can be chosen such that $J\psi \equiv 1$.*

From [7, Proposition 1.2] it follows that A Fatou-Bieberbach domain $\Omega \subset \mathbb{C}^n$ is Runge in $\mathbb{C}^n$ if and only if the associated Fatou-Bieberbach map $F: \mathbb{C}^n \to \Omega$ is a locally uniform limit of holomorphic automorphisms of $\mathbb{C}^n$. Theorem 3.8 demonstrates that an attracting basin associated with a holomorphic automorphism of $\mathbb{C}^n$ is a Runge domain within $\mathbb{C}^n$. Notably, examples of non-Runge Fatou-Bieberbach domains can be found in [13,12]. The following is a corollary of Theorem 3.8 and Theorem 3.2

Corollary 3.9. *Let $F \in Aut(\mathbb{C}^n)$ with fixed point p and Ω be the basin of attraction of F at the point p. Then any biholomorphism $\Phi: \Omega \to \Phi(\Omega)$ with $\Phi(\Omega)$ being Runge can be approximated by elements of $Aut(\mathbb{C}^n)$.*

Proof. Since from Theorem 3.8, it follows that there exists a biholomorphism $G: \mathbb{C}^n \to \Omega$ where Ω is Runge, by Theorem 3.2, it follows that G is a limit of automorphisms of $\mathbb{C}^n$. Therefore, using [7, Proposition 1.2(a)], we obtain that G^{-1} can also be approximated by elements of $\mathrm{Aut}(\mathbb{C}^n)$. If $\Phi: \Omega \to \Phi(\Omega)$ is a biholomorphism with $\Phi(\Omega)$ being Runge then $\Phi \circ G: \mathbb{C}^n \to \Phi(\Omega)$ is a biholomorphism with $\Phi(\Omega)$ is Runge. Hence, again from Theorem 3.2, it follows that $\Phi \circ G$ is limit of $\mathrm{Aut}(\mathbb{C}^n)$. Here G^{-1} is also a limit of a sequence of automorphisms of $\mathbb{C}^n$. Hence, Φ is also the limit of automorphisms of $\mathbb{C}^n$. □

References

[1] Erik Andersén, Volume-preserving automorphisms of C^n, *Complex Variables Theory Appl.*, **14** (1990), no. 1–4, 223–235.

[2] Erik Andersén and László Lempert, On the group of holomorphic automorphisms of C^n, *Invent. Math.*, **110** (1992), no. 2, 371–388.

[3] Leandro Arosio, Filippo Bracci, and Erlend Fornæss Wold, Solving the Loewner PDE in complete hyperbolic starlike domains of C^N, *Adv. Math.*, **242** (2013), 209–216.

[4] Sanjoy Chatterjee and Sushil Gorai, Approximations on spirallike domains of C^n (2023), available at Arxiv: 2208.11107.

[5] Abdeslem El Kasimi, Approximation polynŏmiale dans les domaines étoilés de C^n, *Complex Variables Theory Appl.*, **10** (1988), no. 2–3, 179–182.

[6] Franc Forstnerič, Stein manifolds and holomorphic mappings, Second edition, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 56, Springer, Cham, 2017. The homotopy principle in complex analysis.

[7] Franc Forstnerič and Jean-Pierre Rosay, Approximation of biholomorphic mappings by automorphisms of C^n, *Invent. Math.*, **112** (1993), no. 2, 323–349.

[8] Ian Graham and Gabriela Kohr, Geometric function theory in one and higher dimensions, Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker Inc., New York, vol. 255, (2003).

[9] H. Hamada, Approximation properties on spirallike domains of C^n, *Adv. Math.*, **268** (2015), 467–477.

[10] Jean-Pierre Rosay and Walter Rudin, Holomorphic maps from C^n to C^n, *Trans. Amer. Math. Soc.*, **310** (1988), no. 1, 47–86.

[11] Gerald Teschl, Ordinary differential equations and dynamical systems, *Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI, vol. 140, (2012).

[12] Erlend Fornæss Wold, Fatou-Bieberbach domains, *Internat. J. Math.*, **16** (2005), no. 10, 1119–1130.

[13] Erlend Fornæss Wold, A Fatou-Bieberbach domain in C^2 which is not Runge, *Math. Ann.*, **340** (2008), no. 4, 775–780.

Quasiconformal extensions of conformal mappings

Bappaditya Bhowmik

Department of Mathematics, Indian Institute of Technology Kharagpur.

E-mail: bappaditya@maths.iitkgp.ac.in

Abstract. In this short survey, we first discuss about quasiconformal mappings and motivation behind studying quasiconformal extensions of conformal mappings. After that, we present some sufficient conditions for quasiconformal extensions. Finally, we focus on a few extremal problems for certain classes of conformal mappings having quasiconformal extensions.

Keywords. Quasiconformal mapping, Quasiconformal extension, Univalent functions.

Mathematics Subject Classification 2010: Primary 30C62; Secondary 30C45.

1. Quasiconformal mappings and quasiconformal extensions

Let $\mathbb{C}$ be the finite complex plane and $\mathbb{D}$ be the open unit disk. The complement of the closed unit disk $\overline{\mathbb{D}}$ will be denoted by $\mathbb{D}^*$. Let $\widehat{\mathbb{C}}$ be the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Let Ω_1 and Ω_2 be domains in $\widehat{\mathbb{C}}$. A sense-preserving homeomorphism $f : \Omega_1 \xrightarrow{onto} \Omega_2$ is called k-quasiconformal, $k \in [0, 1)$, if it has locally L^2-derivatives on $\Omega_1 \backslash \{f^{-1}(\infty), \infty\}$ (in the sense of distribution) satisfying

$$|f_{\bar{z}}| \leq k|f_z|, \quad \text{almost everywhere on } \Omega_1,$$

where, $f_z := \partial f/\partial z$ and $f_{\bar{z}} := \partial f/\partial \bar{z}$. The quantity $\mu_f := f_{\bar{z}}/f_z$ is called the complex dilatation of f. If f is k-quasiconformal, then it is also called K-quasiconformal, where, $K := (1 + k)/(1 - k) \in [1, \infty)$. If $k = 0$ or $K = 1$, then by virtue of the Weyl's lemma, f will be conformal on Ω_1. The simplest example of a quasiconformal mapping which is not conformal, is the affine mapping

$$f(z) = az + b\bar{z}, \;\; z \in \mathbb{C} \quad (a, b \in \mathbb{C}, 0 < |b| < |a|).$$

It can be shown that f maps the unit circle $\partial\mathbb{D} := \{z \in \mathbb{C} : |z| = 1\}$ onto an ellipse whose major and minor axes are given by $|a| + |b|$ and $|a| - |b|$ respectively. Hence here, f is a $|b|/|a|$-quasiconformal or, $\frac{|a|+|b|}{|a|-|b|}$-quasiconformal. For another example, we consider the function

$$f(z) = |z|^{\alpha-1}z, \;\; z \in \mathbb{C}, \;\; \alpha \in (0, 1).$$

Here, f maps $\mathbb{D}$ onto itself and it is $\frac{1-\alpha}{1+\alpha}$-quasiconformal or, a $1/\alpha$-quasiconformal map. A key result in understanding planar quasiconformal mappings is the measurable Riemann mapping theorem (henceforth, this theorem will be abbreviated as **MRMT**) of Ahlfors and Bers. We first state this theorem. Let $L^\infty(\mathbb{C})$ be the set of all essentially bounded measurable functions defined in $\mathbb{C}$ and $\mu \in L^\infty(\mathbb{C})$ with $\|\mu\|_\infty < 1$. For every such μ in the open unit ball of $L^\infty(\mathbb{C})$, there exists a q.c. mapping $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ satisfying the Beltrami partial differential equation

$$\frac{\partial f}{\partial \bar{z}} = \mu(z)\frac{\partial f}{\partial z} \text{ for a.e. } z \in \mathbb{C}. \tag{1.1}$$

This map f is a homeomorphism of $\widehat{\mathbb{C}}$ onto itself and it is determined by the Beltrami coefficient μ, uniquely up to post compositions of Moebius transformations of $\widehat{\mathbb{C}}$. Now, one may inquire about the Beltrami PDE (1.1) in a domain $D \neq \widehat{\mathbb{C}}$, for example, in $\mathbb{D}$. By a result of A. Mori (c.f. [25, Theorem 4]), if $w = f(z)$ be a k-quasiconformal mapping of $\mathbb{D}$ onto $\mathbb{D}$, then f can be extended to a topological mapping of the closed disc $\overline{\mathbb{D}}$ onto $\overline{\mathbb{D}}$. Thus, by this result, every q.c. automorphism $f : \mathbb{D} \to \mathbb{D}$ extends homeomorphically to $\partial\mathbb{D}$ and hence, it extends to a q.c. map of $\widehat{\mathbb{C}}$ by the formula

$$f(z) := \frac{1}{\overline{f(1/\bar{z})}}, \;\; z \in \mathbb{D}^*.$$

A little computation will show that the Beltrami coefficient μ of this extension satisfies

$$\mu(z) = \left(\frac{z^2}{\bar{z}^2}\right)\overline{\mu(1/\bar{z})}. \tag{1.2}$$

Now, if $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a q.c. map solving the Beltrami PDE (1.1) with μ satisfying (1.2), then $G(z) := 1/\overline{F(1/\bar{z})}$ is another solution with the same Beltrami coefficient. Thus, by the MRMT, to every μ in the open unit ball of $L^\infty(\mathbb{D})$, there corresponds a q.c. automorphism $f : \mathbb{D} \to \mathbb{D}$ determined uniquely up to Moebius transformations of $\mathbb{D}$.

It is sometime useful to work on the exterior of the unit disc $\mathbb{D}$, i.e., in $\mathbb{D}^*$. Now, for every μ in the open unit ball of $L^\infty(\mathbb{D}^*)$, there corresponds a unique quasiconformal automorphism $f : \mathbb{D}^* \to \mathbb{D}^*$ such that $f(\infty) = \infty$ and $f(1) = 1$. In order to prove this, consider:

$$\frac{\partial F}{\partial \bar{z}} = \mu_1(z)\frac{\partial F}{\partial z}, z \in \widehat{\mathbb{C}}$$

with $F(\infty) = \infty$, $F(1) = 1$, $F(0) = 0$, where

$$\mu_1(z) = \begin{cases} \mu(z), & z \in \mathbb{D}^* \\ \left(\frac{z^2}{\bar{z}^2}\right)\overline{\mu(1/\bar{z})}, & z \in \mathbb{D}. \end{cases}$$

Now, take $f = F|_{\mathbb{D}^*}$.

Let $\mathcal{A}$ be the class of all analytic functions f on $\mathbb{D}$ with the standard normalizations $f(0) = 0 = f'(0) - 1$, and $\mathcal{S}$ be the class of all univalent functions in $\mathcal{A}$. Hence, each function $f \in \mathcal{A}$ has the following Taylor series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1.3}$$

For every μ in the open unit ball of $L^\infty(\mathbb{D}^*)$, there corresponds a conformal map $f \in \mathcal{S}$. To see this, we consider the Beltrami PDE

$$\frac{\partial F}{\partial \bar{z}} = \mu_2(z)\frac{\partial F}{\partial z}$$

with $F(\infty) = \infty$, $F(0) = 0$, $\frac{\partial F}{\partial z}|_{z=0} = 1$, where

$$\mu_2(z) = \begin{cases} \mu(z), & z \in \mathbb{D}^* \\ 0, & z \in \mathbb{D}. \end{cases}$$

Therefore, we now let $f = F|_{\mathbb{D}}$. Hence, these functions are conformal on $\mathbb{D}$ and have quasiconformal extensions on the exterior of $\mathbb{D}$. In this note, we mainly focus on the conformal maps which are defined on a subdomain of the complex plane, but admit quasiconformal extensions to $\mathbb{C}$ or to $\widehat{\mathbb{C}}$. We now present here a precise definition of this extension.

Definition 1.1. *Let $k \in [0, 1)$. We say that a univalent (holomorphic or meromorphic) function f defined on a domain $\Omega \subset \widehat{\mathbb{C}}$ admits a k-quasiconformal extension to $\widehat{\mathbb{C}}$ if there exists a k-quasiconformal mapping $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $F|_\Omega = f$. Moreover, a holomorphic function $f : \Omega \to \mathbb{C}$, ($\Omega \subset \mathbb{C}$), is said to admit a k-q.c. extension to $\mathbb{C}$ if there exists a k-q.c. mapping $F : \mathbb{C} \to \mathbb{C}$ such that $F|_\Omega = f$.*

Using the removability property of quasiconformal mappings, one can conclude that f is k-q.c. extendible to $\mathbb{C}$ if and only if it admits a k-q.c. extension $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with $F(\infty) = \infty$. Due to this fact, q.c.-extendibility of a function f to $\mathbb{C}$ is, in fact, a bit stronger condition than the q.c.-extendibility to $\widehat{\mathbb{C}}$. Normalized holomorphic univalent maps defined on $\mathbb{D}$ having quasiconformal extensions to $\widehat{\mathbb{C}}$ play an important role in Teichmüller theory as they can be identified with the elements of the universal Teichmüller space (see e.g. [24, chap. III]). This is one of the main reasons to study quasiconformal extensions. We will see various examples of conformal mappings that have quasiconformal extensions in our discussion. But, it is interesting to note that there are many conformal maps that do not admit any quasiconformal extensions. For example, the Koebe function $z/(1-z)^2$ maps the unit disk $\mathbb{D}$ onto the slit domain $\mathbb{C}\backslash(-\infty, -1/4]$. This function has no quasiconformal extension to $\mathbb{C}$. The Joukowsky transformation $f(z) = z + 1/z$, $z \in \mathbb{D}$ and the map $f(z) = z + z^2/2$, $z \in \mathbb{D}$ also do not admit any quasiconformal extension. Therefore, it is important to have sufficient conditions for quasiconformal extensions of conformal mappings.

2. Sufficient conditions for quasiconformal extensions

Let $\mathcal{S}_k$ be the class of all functions in $\mathcal{S}$ that have k-q.c. extensions to $\mathbb{C}$. Ahlfors and Weill (c.f. [1]) proved a quasiconformal analogue of the famous Nehari's univalence criteria involving the Schwarzian derivative S_f of a locally univalent function f, where

$$S_f(z) := \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2, z \in \mathbb{D}.$$

They proved that, if f is analytic and locally univalent in $\mathbb{D}$ and

$$|S_f(z)| \leq 2k(1-|z|^2)^{-2}, \quad z \in \mathbb{D},$$

then $f \in \mathcal{S}_k$. Conversely, if $f \in \mathcal{S}_k$ (Kuhnau, 1969; Lehto, 1971), then

$$|S_f(z)| \leq 6k(1-|z|^2)^{-2}, \quad \text{for all } z \in \mathbb{D}.$$

In 1972, Becker proved a criterion for a quasiconformal extension involving the pre-Schwarzian derivative of a locally univalent function. Indeed, he established

Theorem 1 (Becker ([4])). *If $\mathcal{A} \ni f$ is locally univalent and*

$$(1-|z|^2)|zf''(z)/f'(z)| \le k, \quad z \in \mathbb{D},$$

for some $k \in (0,1)$, then $f \in \mathcal{S}_k$ and f has k-quasiconformal extension of the form

$$F(z) = \begin{cases} f(z), & z \in \mathbb{D} \\ f(1/\overline{z}) + (z - 1/\overline{z})f'(1/\overline{z}), & z \in \overline{\mathbb{D}^*} := \{z : |z| \ge 1\}. \end{cases}$$

Later, in 1974, Ahlfors improved the Becker's criterion as following: If $f \in \mathcal{A}$ be locally univalent and there exists a constant $c \in \mathbb{C}$ with $|c| \le k$ such that

$$\left|c|z|^2 + (1-|z|^2)zf''(z)/f'(z)\right| \le k, \quad z \in \mathbb{D},$$

for some $k \in (0,1)$, then $f \in \mathcal{S}_k$. In 1984, Brown([9]) established criteria for quasiconformal extensions of starlike, convex, and analytic functions with bounded derivative, using geometric characterization of the image domain. Brown proved that, if $f \in \mathcal{A}$ and

$$\frac{zf'(z)}{f(z)} \prec \frac{1+kz}{1-kz} \quad \text{or} \quad 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+kz}{1-kz},$$

then $f \in \mathcal{S}^* \cap \mathcal{S}_k$, or $f \in \mathcal{C} \cap \mathcal{S}_k$, respectively. Here, $\mathcal{S}^*$ and $\mathcal{C}$ denote the classes of starlike and convex functions on $\mathbb{D}$ respectively. We clarify here that the symbol $\prec$ stands for the well-known notion of *subordination*. The explicit k-quasiconformal extension of f is given below:

$$F(z) = \begin{cases} f(z), & z \in \mathbb{D} \\ |z| f(z/|z|), & z \in \overline{\mathbb{D}^*}. \end{cases}$$

Brown also proved that, if $f \in \mathcal{A}$ with

$$|z^2(\lambda f'(z) - 1)| \le k, \ \ z \in \mathbb{D}, \ \ (\lambda \in \mathbb{C}, \text{ a constant}),$$

then $f \in \mathcal{S}_k$, and has the k-q.c. extension:

$$F(z) = \begin{cases} f(z), & z \in \mathbb{D} \\ f(1/\overline{z}) + (1/\lambda)(z - 1/\overline{z}), & z \in \overline{\mathbb{D}^*}. \end{cases}$$

In [29], Sugawa obtained the above results as an application of holomorphic motions and the λ-Lemma. Quasiconformal extension criteria for functions in the class of strongly starlike functions of order $\alpha \in (0,1)$, denoted by $\mathcal{S}^*(\alpha)$, was obtained by Fait et al. (compare [12]). Let $f \in \mathcal{A}$. It is well-known that, $f \in \mathcal{S}^*(\alpha) \subset \mathcal{S}$ if and only if

$$\left|\arg \frac{zf'(z)}{f(z)}\right| \le \frac{\pi\alpha}{2}, \quad z \in \mathbb{D}.$$

Fait et al. proved that each function $f \in \mathcal{S}^*(\alpha) \subset \mathcal{S}$ admits $\sin(\pi\alpha/2)$-quasiconformal extension onto $\widehat{\mathbb{C}}$, which can be given explicitly as:

$$F(z) = \begin{cases} f(z), & z \in \overline{\mathbb{D}} \\ \frac{(f(e^{i\theta}))^2}{f(1/\overline{z})}, & z \in \overline{\mathbb{D}^*}, \end{cases}$$

where, $f(e^{i\theta})$ is uniquely determined by the equation

$$\arg f(1/\overline{z}) = \arg f(e^{i\theta}), \ \ z \in \mathbb{D}^*.$$

We add here that, Sevodin ([28]) proved q.c. extension criterion for strongly spirallike functions (see also [30]).

In 1972, Becker proved a remarkable result concerning quasiconformal extensions of univalent functions with the help of Loewner chain. Let

$$f_t(z) = f(z,t) := e^t z + \sum_{n=2}^{\infty} a_n(t) z^n, \quad (z,t) \in \mathbb{D} \times [0,\infty), \tag{2.1}$$

be a family of functions. Such a family $f_t(z)$ is called a Loewner chain if $f_t(z)$ is analytic and univalent in $\mathbb{D}$ for each fixed $t \in [0,\infty)$ and satisfies $f_s(\mathbb{D}) \subsetneq f_t(\mathbb{D})$ for $0 \le s < t < \infty$. Pommerenke (see [26, Theorem 6.2]) proved a necessary and sufficient condition for a family of functions $f(z,t)$ to be a Loewner chain. Later, Becker proved the following result:

Theorem 2 (Becker ([4]). *Let*

$$f_t(z) = f(z,t), \ (z,t) \in \mathbb{D} \times [0,\infty)$$

be a Loewner chain such that the Herglotz function p in the Loewner-Kufarev PDE

$$\frac{\partial f(z,t)}{\partial t} = zf'(z,t)p(z,t)$$

lies within the disk $\mathcal{D}(k)$ for all $z \in \mathbb{D}$ and for almost all $t \in [0,\infty)$, where,

$$\begin{aligned} \mathcal{D}(k) &:= \left\{ w \in \mathbb{C} : \left|\frac{w-1}{w+1}\right| \le k \right\} \\ &= \left\{ w \in \mathbb{C} : \left| w - \frac{1+k^2}{1-k^2} \right| \le \frac{2k}{1-k^2} \right\} \subsetneq \{z : \operatorname{Re} z > 0\}; \end{aligned}$$

then for each $t \ge 0$, each function f_t admits a k-quasiconformal extension to $\mathbb{C}$ fixing infinity and such an extension for the initial member f_0 is given by

$$F(z) = \begin{cases} f_0(z) = f(z,0), & z \in \mathbb{D} \\ f(z/|z|, \log|z|), & z \in \overline{\mathbb{D}^*}. \end{cases}$$

The extension F of $f = f_0$ in this result is called a *Becker extension* of $f \in \mathcal{S}$. Using Pommerenke's and Becker's criteria, Hotta (c.f. [16]) constructed suitable Loewner chains to prove the quasiconformal extension criteria for various subclasses of $\mathcal{S}$, which were earlier proved by Brown in [9]. Using this method, the author of this survey and G. Satpati proved the criterion for quasiconformal extension for functions in $\mathcal{U}_\lambda \subset \mathcal{S}$, $(\lambda \in (0,1))$, (see f.i. [8]). We mention here that, $\mathcal{U}_\lambda$ consists of all functions in $\mathcal{A}$ that satisfy

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right| < \lambda$$

for $z \in \mathbb{D}$. In [17], Hotta modified the Loewner chain in (2.1), replacing the leading coefficient $f'(0,t) = e^t$ by any complex valued function $a_1(t)$, such that $|a_1(t)| \to \infty$ as $t \to \infty$. In the same article, Hotta also proved the quasiconformal extension criteria for α-spirallike functions $(-\pi/2 < \alpha < \pi/2)$. A function $f \in \mathcal{A}$ is said to be α-spirallike, if

$$\mathrm{Re}\{e^{-i\alpha} z f'(z)/f(z)\} > 0, \quad z \in \mathbb{D}.$$

For $\alpha = 0$, we have the class of starlike functions. Pommerenke ([26, Chap. 6]) constructed the following Loewner chain for an α-spirallike function:

$$f(z,t) = e^{(1+ia)t} f(e^{-iat} z), \ (a = \tan\alpha), \quad z \in \mathbb{D}, \ t \in [0,\infty).$$

Hotta (see [17]) used this Loewner chain to prove the following: If $f \in \mathcal{A}$ such that $zf'(z)/f(z) \in \mathcal{D}(k,\alpha)$, where $\mathcal{D}(k,\alpha)$ is the following hyperbolic disk in the tilted half-plane $\{z \in \mathbb{C} : \mathrm{Re}\,(e^{-i\alpha} z) > 0\}$:

$$\begin{aligned}\mathcal{D}(k,\alpha) &:= \left\{w \in \mathbb{C} : \left|\frac{w-1}{w+e^{2i\alpha}}\right| \le k\right\} \\ &= \left\{w \in \mathbb{C} : \left|w - \frac{1+e^{2i\alpha}k^2}{1-k^2}\right| \le \frac{2k\cos\alpha}{1-k^2}\right\};\end{aligned}$$

then f is α-spirallike which admits a k-q.c. extension of the form:

$$F(z) = \begin{cases} f(z), & z \in \mathbb{D} \\ |z|^{1+ia} f(z/|z|^{1+ia}), & z \in \overline{\mathbb{D}^*}. \end{cases}$$

We remark here that, the quasiconformal extensibility for an α-Robertson function was also obtained by Hotta and Wang in [18]. We mention here that, a function $f \in \mathcal{A}$ is said to be α-Robertson $(-\pi/2 < \alpha < \pi/2)$, if

$$\mathrm{Re}\left\{e^{-i\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > 0, \quad z \in \mathbb{D}.$$

In [27], Royster studied the problem for univalence of the following two integral transforms;

$$f_\alpha(z) = \int_0^z (f'(\zeta))^\alpha \, \mathrm{d}\zeta \quad \text{and} \quad F_\alpha(z) = \int_0^z \left(\frac{f(\zeta)}{\zeta}\right)^\alpha \mathrm{d}\zeta,$$

where $f \in \mathcal{S}$. Here we note that, the powers in the above two definitions are defined via the branch of $\log f'(\zeta)$ for which $\log f'(0) = 0$. Hotta and Wang obtained criteria for quasiconformal extensions of the above mentioned two integral transforms in [19].

We now turn our attention to the meromorphic case. Let Σ be the class of all meromorphic univalent functions on $\mathbb{D}$ having simple pole at $z = 0$ with residue 1. Therefore, each $f \in \Sigma$ has the Laurent series expansion as

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n, \quad z \in \mathbb{D}. \tag{2.2}$$

Let Σ_k be the class of all functions in Σ that admit k-quasiconformal extensions to $\widehat{\mathbb{C}}$. We first mention here that, a sufficient condition was proved by Krzyż in [22] for a function in Σ to belong in the class Σ_k. This result states that if $f \in \Sigma$ is of the form $f(z) = 1/z + \omega(z), z \in \mathbb{D}$, where ω is an analytic function on $\mathbb{D}$ such that $|\omega'(z)| \le k$ for some $k \in (0,1)$, then $f \in \Sigma_k$. The k-quasiconformal extension of f is given by

$$f(z) = \frac{1}{z} + \omega(1/\overline{z}), \ z \in \overline{\mathbb{D}^*}.$$

Let $\Sigma(p)$ be the class of all meromorphic univalent functions having pole at some point $p \in [0,1)$ with residue 1. Thus, any $f \in \Sigma(p)$ has the Laurent expansion of the following form:

$$f(z) = \frac{1}{z-p} + \sum_{n=0}^{\infty} b_n z^n, \quad z \in \mathbb{D}. \tag{2.3}$$

In [6], we defined the class $\Sigma_k(p)$ as the class of all functions in $\Sigma(p)$ that have k-quasiconformal extensions to $\widehat{\mathbb{C}}$. We proved that, if $f(z) = 1/(z-p) + \omega(z) \in \Sigma(p)$ (ω being analytic in $\mathbb{D}$) such that $|\omega'(z)| \le k(1+p)^{-2}$ for $z \in \mathbb{D}$, then $f \in \Sigma_k(p)$, (compare [6, Theorem 2]). The k-quasiconformal extension of f is given by

$$f(z) = \frac{1}{z-p} + \omega(1/\overline{z}), \ z \in \overline{\mathbb{D}^*}.$$

Next, we discuss about the area distortion by quasiconformal mappings. In 1955, Bojarski first considered the area distortion problem for a general quasiconformal mapping. In 1994, Astala (c.f. [2]) proved the following result: Let $f : \mathbb{D} \to \mathbb{D}$ be a k-quasiconformal mapping with $f(0) = 0$, then

$$|f(E)| \le M(K)|E|^{1/K},$$

for all Lebesgue measurable sets $E \subset \mathbb{D}$, where $K = (1+k)/(1-k) \ge 1$, and the constant $M(K) = 1 + \mathcal{O}(K-1)$, as $K \to 1$. Here, the notation $|\cdot|$ stands for the area of a set in the complex plane. Later, Eremenko and Hamilton (compare [11]) proved the following area distortion inequalities for functions belonging to the class Σ_k.

Theorem 3. *If $f \in \Sigma_k^0$ $(b_0 = 0)$ and E be a Lebesgue measurable subset of $\mathbb{D}$, then*

$$|f(E)| \le \pi^{1-1/K}|E|^{1/K},$$

whenever f is conformal on E, and

$$|f(E)| \le K|E|,$$

whenever f is conformal on $\mathbb{C}\backslash E$. Also, for any arbitrary Lebesgue measurable subset E of $\mathbb{D}$,

$$|f(E)| \le K\pi^{1-1/K}|E|^{1/K}.$$

All the constants in the last three inequalities are best possible.

Astala and Nesi (c.f. [3]) also proved the weighted area distortion inequalities for functions in the class Σ_k^0. The corresponding area distortion inequalities for functions belonging to the class $\Sigma_k^0(p)$ can be found in [7].

3. Extremal problems for conformal mappings having quasiconformal extensions

The problem of estimating bounds for the moduli of the Taylor and the Laurent coefficients for functions in the classes $\mathcal{S}_k$ and Σ_k were well-researched by many eminent mathematicians. We present here a brief overview of this. Let $\mathcal{S}_k^\infty$ consists of all functions in $\mathcal{S}_k$ which fix the point infinity, i.e. $f(\infty) = \infty$. If $f \in \mathcal{S}_k^\infty$ having an expansion of the form (1.3) in $\mathbb{D}$, then for $n = 2$, R. Kühnau (in 1969) proved the sharp estimate $|a_2| \le 2k$, where equality holds for the following function:

$$f_k(z) = \begin{cases} \frac{z}{(1-kz)^2}, & z \in \mathbb{D} \\ \frac{z\bar{z}}{(\sqrt{\bar{z}}-k\sqrt{z})^2}, & z \in \overline{\mathbb{D}^*}. \end{cases} \tag{3.1}$$

It is easy to verify that $f_k \in \mathcal{S}_k$ with $|a_2| = 2k$. In 1968, Krushkal ([21]) gave an asymptotic bound for the Taylor coefficients in the following form:

$$|a_n| \le \frac{2k}{n-1} + \mathcal{O}(k^2), \quad k \to 0,$$

for all $n \ge 2$, where, the ratio $\mathcal{O}(k^2)/k^2$ is uniformly bounded for all $k \le k_0 < 1$, where k_0 is a fixed number. The equality $|a_n| = 2k/(n-1)$, for $n \ge 3$, occurs for the mapping

$$\begin{aligned} f_{k,n-1}(z) &:= (f_k(z^{n-1}))^{1/(n-1)} \\ &= z + (2k/(n-1))z^n + a_{2n-1}z^{2n-1} + \cdots, \quad z \in \mathbb{D}, \end{aligned}$$

where, f_k is given by (3.1). In 1995, Krushkal ([20]) also claimed that:

$$|a_n| \le 2k/(n-1), \tag{3.2}$$

for $0 < k \le 1/(n^2+1)$ and $n \ge 3$. Let S_k^B be the class of all $f \in \mathcal{S}$ admitting Loewner's representation with the Herglotz function p normalized by $p(0,t) = 1$ a.e. $t \ge 0$ and satisfying $p(\mathbb{D}, t) \subset \mathcal{D}(k)$. It was shown that $S_k^B \subsetneq \mathcal{S}_k$, (see [14]). Gumenyuk and Hotta proved (see [15]) that the following sharp estimate holds for functions in S_k^B:

$$|a_3| \le k(1 + e^{1-1/k}(1+k)).$$

This disproves a result of S. Krushkal (compare with (3.2)) which was proved in [20], at least for a_3. In 1976, Lehto proved asymptotic bounds for the Laurent coefficients for functions in Σ_k (see f.i. [24, p. 74]). Indeed, Lehto obtained that, if $f \in \Sigma_k^0$ having expansion of the form (2.2), then

$$|b_n| \le \frac{2k}{n+1} + \mathcal{O}(k^2), \quad n \ge 1.$$

If

$$f_n(z) = \begin{cases} \left(z^{-(n+1)/2} + kz^{(n+1)/2}\right)^{2/(n+1)}, & z \in \mathbb{D}, \\ \left(z^{-(n+1)/2} + k\bar{z}^{-(n+1)/2}\right)^{2/(n+1)}, & z \in \overline{\mathbb{D}^*}, \end{cases}$$

then $f_n \in \Sigma_k^0$ having complex dilatation

$$\mu_{f_n}(z) = k(z/\bar{z})^{(n+3)/2}, \quad z \in \overline{\mathbb{D}^*}$$

with $|b_n| = 2k/(n+1)$. Lehto also established the exact estimate $|b_n| \le 2k/(n+1)$, for $n = 1, 2$. It is worth to mention here that, the exact coefficient estimates of $|a_n|$ for $\mathcal{S}_k$ and of $|b_n|$ for Σ_k are still open for all $n \ge 3$.

In this survey, finally, we briefly discuss on area theorems for meromorphic univalent functions having quasiconformal extensions. In [23], Lehto refined the Bieberbach-Gronwall area theorem for functions in Σ_k. This result states that if $f \in \Sigma_k$ having expansion (2.2) in $\mathbb{D}$, then

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq k^2,$$

where, equality holds if and only if

$$f(z) = \frac{1}{z} + b_0 + b_1 z, \quad z \in \mathbb{D}, \quad \text{with } |b_1| = k.$$

Moreover, its k-quasiconformal extension is given by setting

$$f(z) = \frac{1}{z} + b_0 + \frac{b_1}{\bar{z}}, \ \ z \in \overline{\mathbb{D}^*}.$$

Considering a nonzero simple pole in the open unit disc, Chichra (c.f. [10]) proved the following Area Theorem for functions in the class $\Sigma(p)$, $p \in (0,1)$. Let $f \in \Sigma(p)$ have expansion of the form (2.3), then

$$\sum_{n=1}^{\infty} n|b_n|^2 \ \leq \ \frac{1}{(1-p^2)^2},$$

where, equality holds for the function

$$f(z) = \frac{1}{z-p} + b_0 + \frac{z}{1-p^2}, \ z \in \mathbb{D}.$$

Inspired by the Lehto's Area theorem for functions in Σ_k, the following Area theorem for the class $\Sigma_k(p)$ was obtained in [6, Theorem 1]:

Theorem 4. *Let $0 \leq k < 1$ and $0 \leq p < 1$. Suppose that $f \in \Sigma_k(p)$ has expansion of the form (2.3). Then*

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq \frac{k^2}{(1-p^2)^2}.$$

Here, equality holds if and only if f is of the form

$$f(z) = \frac{1}{z-p} + b_0 + \frac{b_1 z}{1-pz}, \quad z \in \mathbb{D},$$

where, b_0 and b_1 are constants with $|b_1| = k$. Moreover, a k-quasiconformal extension of this f is given by setting

$$f(z) = \frac{1}{z-p} + a_0 + \frac{b_1}{\bar{z}-p}, \qquad z \in \overline{\mathbb{D}^*}.$$

We urge interested readers to go though the articles [5,6] for more details and other related open problems for functions in the class $\Sigma_k(p)$.

Acknowledgement

This survey article is a slight modification of an extended abstract of a lecture presented by the author at the International Workshop on Geometric Function Theory 2023 (IWGFT 2023) which was held at IIT Madras from 18$^{\text{th}}$ August to 20$^{\text{th}}$ August, 2023. The author would like to thank Prof. Toshiyuki Sugawa for his careful reading and valuable suggestions during preparation of this article.

References

[1] L. Ahlfors and G. Weill, A uniqueness theorem for Beltrami equations, *Proc. Amer. Math. Soc.*, **13**, (1962), 975–978.

[2] K. Astala, Area distortion of quasiconformal mappings, *Acta Math.*, **173**, no. 1, (1994), 37–60.

[3] K. Astala and V. Nesi, Composites and quasiconformal mappings: new optimal bounds in two dimensions, *Calc. Var. Partial Differential Equations*, **18**, no. 4, (2003), 335–355.

[4] J. Becker, Conformal mappings with quasiconformal extensions, Aspects of contemporary complex analysis (Proc. NATO Adv. Study Inst., Univ. Durham), Academic Press, (1980), 37–77.

[5] B. Bhowmik and G. Satpati, On some results for a class of meromorphic functions having quasiconformal extension, *Proc. Indian Acad. Sci. Math. Sci.*, **128**, no. 5, Paper No. 61, (2018), 16 pp.

[6] B. Bhowmik, G. Satpati and T. Sugawa, Quasiconformal extension of meromorphic functions with nonzero pole, *Proc. Amer. Math. Soc.*, **144**, no. 6, (2016), 2593–2601.

[7] B. Bhowmik and G. Satpati, Area distortion under meromorphic mappings with nonzero pole having quasiconformal extension, *J. Korean Math. Soc.*, **56**, no. 2, (2019), 439–455.

[8] B. Bhowmik and G. Satpati, Loewner chain and quasiconformal extension of some classes of univalent functions, *Complex Var. Elliptic Equ.*, **65**, no. 4, (2020), 544–557.

[9] J. E. Brown, Quasiconformal extensions for some geometric subclasses of univalent functions, *Internat. J. Math. & Math. Sci.*, **7**, no. 1, (1984), 187–195.

[10] P. N. Chichra, An area theorem for bounded univalent functions, *Proc. Cambridge Philos. Soc.*, **66**, (1969), 317–321.

[11] A. Eremenko and D. H. Hamilton, On the area distortion by quasiconformal mappings, *Proc. Amer. Math. Soc.*, **123**, no. 9, (1995), 2793–2797.

[12] M. Fait, J. G. Krzyż and J. Zygmunt, Explicit quasiconformal extensions for some classes of univalent functions, *Comment. Math. Helv.*, **51**, (1976), 279–285.

[13] F. W. Gehring and E. Reich, Area distortion under quasiconformal mappings, *Ann. Acad. Sci. Fenn. Ser. AI*, no. 388, (1966), 1–14.

[14] P. Gumenyuk and I. Prause, Quasiconformal extensions, Loewner chains, and the λ-Lemma, *Analysis and Mathematical Physics*, **8**, (2018), 621–635.

[15] P. Gumenyuk and I. Hotta, Univalent functions with quasiconformal extensions: Becker's class and estimate of te third coefficeint, *Proc. Amer. Math. Soc.*, **148**, no. 9, (2020), 3927–3942.

[16] I. Hotta, Explicit quasiconformal extensions and Löwner chains, *Proc. Japan Acad. Ser. A Math. Sci.*, **85**, no. 8, (2009), 108–111.

[17] I. Hotta, Löwner chains with complex leading coefficient, *Monatsh. Math.*, **163**, no. 3, (2011), 315–325.

[18] I. Hotta and Li-Mei Wang, Boundedness, univalence and quasiconformal extension of Robertson functions, *Indian J. Pure Appl. Math.*, **42**, no. 4, (2011), 239–248.

[19] I. Hotta and Li-Mei Wang, Quasiconformal extendability of integral transforms of Noshiro-Warschawski functions, *Rocky Mountain J. Math.*, **47**, no. 1, (2017), 185–204.

[20] S. Krushkal, Exact coefficient estimates for univalent functions with quasiconformal extension, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **20**, no. 2, (1995), 349–357.

[21] S. Krushkal, Some extremal problems for schlicht analytic functions, *Soviet Math. Dokl.*, **9**, (1968).

[22] J. G. Krzyż, Convolution and quasiconformal extension, *Comment. Math. Helv.*, **51**, (1976), 99–104.

[23] O. Lehto, Schlicht functions with a quasiconformal extension, *Ann. Acad. Sci. Fenn. Ser. A. I.*, **500**, (1971), 1–10.

[24] O. Lehto, Univalent Functions and Teichmüller Spaces, Springer-Verlag, New York, (1987).

[25] A. Mori, On quasi-conformality and pseudo-analyticity, *Trans. Amer. Math. Soc.*, **84**, (1957), 56–77.

[26] Ch. Pommerenke, Univalent functions, Vandenhoeck & Ruprecht, Göttingen, (1975).

[27] W. C. Royster, On the univalence of a certain integral, *Michigan Math. J.*, **12**, (1965), 385–387.

[28] M. A. Sevodin, Univalence conditions in spiral-shaped domains, *Tr. Semin. Kraev. Zad.*, **23**, (1987), 193–200 (in Russian).

[29] T. Sugawa, Holomorphic motions and quasiconformal extensions, *Ann. Univ. Mariae Curie-Skłodowska Sect. A*, **53**, (1999), 239–252.

[30] T. Sugawa, Quasiconformal extension of strongly spirallike functions, *Comput. Methods Funct. Theory*, **12**, no. 1, (2012), 19–30.

Twenty Ninth ICFIDCAA-2023 Conference Report

The 29th "International Conference on Finite or Infinite Dimensional Complex Analysis and Applications" was being hosted by the Department of Mathematics, Pondicherry University from 21st to 25th August 2023. The five-day conference was graced by international delegates from Russia, Japan, South Korea, Serbia, Croatia, Malaysia and USA and over 100 Indian participants.

Followed by the inaugural ceremony, the Keynote Speaker Prof. S. Ponnusamy delivered a lecture on "Landau-Bloch Theorems for Analytic, Meromorphic, and Harmonic Functions", followed by a talk of Prof. Sudeb Mitra, City University of New York on "Tame Quasiconformal Motions".

An invited talk by Prof. Allu, was delivered. In addition, many short talks and paper presentation sessions were held after the invited talks.

The second day commenced with the plenary talks by Prof. Armen Sergev "On Mathematical Problems in the Theory of Topological Insulators"; Prof. Toshiyuki Sugawa, on "Quasiconformal distortion of hyperbolic distances"; Prof. Jaydeb Sarkar on "Commutant lifting theorem on the polydisc" ; and Prof. Kaushal Verma, on "A report on the Grauert metric". The invited Talks were presented by Prof. Krishnendu Gongopadhyay, and Prof. Sanjeev Singh. The parallel paper presentation sessions were held after the invited talks.

The third day began with the Plenary Talks by Prof. Solodov Aleksei, on "Extremal problems on classes of holomorphic self-maps of a disc with fixed points" and Prof. Tibor K. Pogany, "On Rice–Middleton model of probability distribution of sinusoidal signal combined with Gaussian noise". The plenary talks were followed by Invited Talks by Prof. Boo Rim Choe, Prof. Swadesh Kumar Sahoo, Prof. Jugal K Prajapat, Prof. Ritu Agarwal. In addition to the invited talks, the paper presentation sessions were held which included presentations by professors and research scholars.

The fourth day began with the Plenary Talks by Prof. Hiroshi Yanagihara, on "Loewner Theory on Analytic Universal Covering Mappings" and Prof. Tirthankar Bhattacharyya, on "Approximation by inner functions - a Hilbert space approach". The invited talks were given by Prof. See Keong Lee, Prof. Vesna Todorcevic, and Prof. A. Sairam Kaliraj.

The fifth day invited talks were given by Prof. Aleksandr Komlov, and Prof. S. N. Fathima. A valedictory program was held and Prof. Gurmeet Singh, Honourable Vice chancellor, Pondicherry University delivered the presidential address and the conference was official concluded.

The conference was supported by National Board of Higher Mathematics, Department of Science and Technology, New Delhi and Pondicherry University.

Dr. Rakesh Kumar Parmar
Convener, ICFIDCAA-2023
Pondicherry University

Report on the International Conference on Mathematics and Computing-ICMC (January 04–07, 2024) and a Pre-Conference National Symposium on "Advanced Mathematical Methods" (January 02–03, 2024)

The 10th edition of ICMC2024 and a Pre-Conference National Symposium on "Advanced Mathematical Methods was organized by Kalasalingam Academy of Research and Education jointly with academic sponsorship of Ramanujan Mathematical Society–RMS, Cryptology Research Society of India-CRSI and Society for Electronics Transactions and Security-SETS. The Symposium was sponsored partly by National Board of Higher Mathematics-NBHM and the conference was sponsored partly by Defence Research and Development Organization-DRDO. There were eighteen invited talks delivered by eminent professors from various countries like USA, Australia, Turkey and India. The conference received 286 research articles from which 40 articles were carefully selected and will be published with SPRINGER in two volumes. Also, a panel discussion on the "Importance of Mathematics and Computing for all round development of our Nation" was held on 5th January 2024. The experts drawn from Chennai Mathematical Institute-CMI, IIT Ropar, Ramanujan Institute of Advanced Study in Mathematics-RIASM of University of Madras, Society for Electronics Transactions and Security-SETS and Madras School of Economics gave the onus of training to the faculty members and research scholars. Accordingly the experts have delivered special lectures and imparted hands on training in problem solving. The conference was attended by 200 participants from 10 countries and the symposium was attended by 100 participants from all over India. The programme has received an overwhelming response and earned an excellent feedback from all types of delegates. List of Invited Speakers for the Symposium: Drs. Clare D'Cruz, G. P. Youvaraj, Tapas Chatterjee, Gautham Sekar, Prem Lakshman Das. List of Invited Speakers for the Conference is: Drs. Elisa Bertino, Bhavani Thuraisingham, Muriel Medard, Ramamohanarao Kotagiri, Mohammad Obaidat, Sedat Akleylek, Ekrem Savas, Clare D' Cruz, R. K. Sharma, Madhumangal Pal, Jaya N Iyer, Arvind Ayyar, Tanmoy Som, A. K. B. Chand, Mark Sepanski, Vishnu Pendyala, and Tanmay Basak.

Professors Debasis Giri, Saminathan Ponnusamy, and Yegnanarayanan Venkataraman
ICMC-2024

Report on ICAADM 2024

10th International Conference on Applied Analysis and Discrete Mathematics (ICAADM 2024) was organized by Department of Mathematics, The Gandhigram Rural Institute (Deemed to be University), Gandhigram, Tamil Nadu, India during January 22–24, 2024. Prof. Uthayakumar, Co-Convener, delivered the welcome address. Dr. R. Rajkumar, Convener of this conference outlined the mechanics of this conference. Prof. S. Ponnusamy, IIT Madras delivered the presidential address. Prof. L. Rathakrishnan, Registrar (in charge), Prof. M. G. Sethuraman, Dean, School of Sciences, GRI have also addressed the gathering. Professors P. Kandaswamy (Coimbatore), Mahyar Mahinzaeim (Germany), Kurunathan Ratnavelu, (Malaysia), Kohilah Miundy, (Malaysia) A. K. Nandakumaran, IISc, Awdhesh Prasad, (Delhi), Rajesh Kannan, IIT Hyderabad gave invited/keynote addresses. Prof. P. Balasubramaniam (Co-Convener) delivered the vote of thanks. The book of abstracts of this conference was released by the dignitaries. About 120 participants both from India and abroad participated in this conference. More than hundred papers have been presented in this conference. This conference was supported by DST, SERB, CSIR and UGC.

Report on Workshop on Algebra, Analysis and Number Theory-2024

The Department of Mathematics at The Gandhigram Rural Institute (Deemed to be University), Gandigram, successfully conducted a **three-day workshop on Algebra, Analysis, and Number Theory (WAANT-2024) from January 25–27, 2024**. The event garnered participation of students from various colleges in and around Dindigul and Madurai. This workshop was to guide third year B.Sc., mathematics students to explore the new way of understanding the mathematical concepts. There were 45 students participated in the workshop; 30 were from outside colleges, and the remaining 15 from the host Institute.

The main idea of this workshop was to train students as how to learn mathematics so that understanding is easy and enjoyable. Topics from Algebra, Analysis and Number theory were discussed. Professors R. Balasubramanian (IMSc) - Number Theory, K. N. Raghavan (IMSc) - Algebra, P. Veeramani (IITM) - Real Analysis, G. P. Youvaraj (RIASM) - Complex Analysis were the resource persons. There were 12 sessions, 4 each day. From the feedbacks obtained, it was evident that students have enjoyed new ways of learning and understanding mathematics. They expressed desire for more such programmes.

This programme was supported by The National Academy of Sciences (NASCI), Chennai Mathematical Institute (CMI), The Institute of Mathematical Sciences (IMSc) and Gandhigram Rural Institute-Deemed to be University.

International Workshop on Geometric Function Theory (IWGFT 2023)

August 18–20, 2023.

IIT Madras jointly with Forum d' Analystes, Chennai, hosted an International Workshop on Geometric Function Theory (IWGFT 2023) during August 18–20, 2023, attracting more than 114 participants, including the speakers from Japan, Croatia, Russia, Serbia, China (online), and different parts of India.

The workshop was, for the most part, focused on the inspiring branches of Complex Analysis, namely, univalent functions, hyperbolic-type geometry, function spaces, special functions, harmonic and quasiconformal mappings, and several complex variables. The primary emphasis of the workshop was on training Ph.D. students, postdoctoral fellows, and other young researchers, and promoting collaborative research involving all the participants. The three-day workshop had several problem discussion sessions from which participants benefited by gaining professional knowledge and skills. Sixteen lectures were organized during the workshop. Details are available at https://sites.google.com/view/iwgft2023/schedule.

For the students' benefit, the full length lecture materials are being published in the *Mathematics News Letter* published by *Ramanujan Mathematical Society* in three different issues. It is our hope and intent that these materials will inspire people working in the focused area of the workshop; indeed, some of the talks also listed open problems in which readers can engage in, and collaborate with workshop participants.

The organizing committee is thankful to the following agencies for their financial support to organize the IWGFT 2023 successfully:

(i) Council of Scientific and Industrial Research (CSIR), New Delhi

(ii) National Board for Higher Mathematics (NBHM), DAE, Mumbai

(iii) Science and Engineering Research Board (SERB), New Delhi

Organizers of the workshop:
Professors R. Balasubramanian, S. Ponnusamy, S. K. Sahoo.

Report on Teachers' Enrichment Workshop on Linear Algebra, Real Analysis and Topology

Teachers' Enrichment Workshop, for college teachers, Research scholars was conducted at Department of Mathematics, Mepco Schlenk Engineering College, Sivakasi during November 27–December 02, 2023. Drs. Arindama Singh, IITM-Linear Algebra, S. Ponnusamy, IITM-Real Analysis, and G. P. Youvaraj, RIASM-Topology, were the resource persons. The workshop focused on the fundamentals and problem solving aspects. There were 30 college teachers (some from Kerala, Karnataka), and 5 research scholars. On each day, there were three 90 minutes lectures followed by a 60 minutes tutorial session in problem solving. The host institution provided free boarding and lodging for resource persons and participants from for outside the local area. Based on the feedback from direct and google form, participants had benefited in improving mathematical skills.

The workshop was supported by National Center for Mathematics, and Mepco Schlenk Engineering College, Sivakasi, Tamil Nadu.

Dr. G. P. Youvaraj	Dr. R. Ratha Jeyalakshmi,
Academic Convener	Local Organizer
Ramanujan Institute,	Mepco Schlenk Engg. College,
University of Madras	Sivakasi

Report on 38th Annual Conference of the Ramanujan Mathematical Society

The annual conference is one of the largest and most important activities of the Ramanujan Mathematical Society (RMS). The 38th Annual Conference of the RMS was held at IIT Guwahati on December 22–24, 2023. There were a total of 452 registered participants and a total of 13 symposia. The topics addressed in these symposia consisted of almost all main branches of mathematics; and notably Women in Mathematics and History of Mathematics in India. The total number of symposia talks was 180 and the conference also had five plenary talks along with a public lecture on the history of mathematics. On top of all, the annual conference hosted a total of 145 contributed talks on various topics. This time, approximately 26% of the participants were female. An estimated quarter of the participants come from the north eastern region. Seven out of thirteen organisers of symposia are women.

This annual event was generously supported by the Indian National Science Academy (INSA), the History of Mathematics in India (HoMI), the National Board of Higher Mathematics (NBHM), and IIT Guwahati.

Jaydeb Sarkar January, 2023
Academic Secretary, Ramanujan Mathematical Society

Sukanta Pati
Convenor, Local Organising Committee, IIT Guwahaty

Announcement of IITH Institute Postdoctoral Fellowship (IITH-IPDF) (Call is Open now)

Aim: To attract and support early-career researchers with exceptional academic potential to pursue independent research under the mentorship of esteemed faculty.

No. of Fellowships and Stipend: 50; 75,000 INR p.m.
Duration: One year, with the possibility of renewal for an additional year based on satisfactory performance.
Accommodation: On-campus accommodation will be provided. No HRA will be applicable, in case the Fellow does not wish to stay inside campus.
Submission of Applications: Feb 1, 2024 to Feb 26, 2024 (Online mode only; by Applicants directly while attaching endorsements from you)
Announcement of results: March 31, 2024
Joining Date: April 15, 2024
Eligibility: Applicants must hold a Ph.D. degree in a relevant field from a recognized university (IITH PhD graduates are NOT eligible). Those who submitted the thesis are also eligible.
Research Focus: Open to all disciplines
Mentorship: Each fellow will be assigned a faculty mentor within their field of expertise as per the proposal submitted. **One faculty member can mentor only one applicant who is selected for the program in this round.**
Endorsements: Each faculty member can provide **up to two endorsements** for potential applicants. In case, more than two endorsements are received from the same mentor, the first two will be only considered.

More details and Application Form: https://iith.ac.in/news/2024/01/31/IITH-IPDF-2024/ OR IIT Hyderabad

---

Details of Workshops/Conferences in India

For details regarding Mathematics Training and Talent Search Programme
Visit: https://mtts.org.in/programme/mtts2021/

For details regarding Annual Foundation Schools, Advanced Instructional Schools, NCM Workshops, Instructional Schools for Teachers, Teacher's Enrichment Workshops
Visit: https://www.atmschools.org/

Name: 26^{th} Annual Conference of The Society of Statistics, Computer and Applications (SSCA) International Conference on Emerging Trends of Statistical Sciences in AI and its Applications (ETSSAA-2024).
Date: February 26, 2024–February 28, 2024
Venue: Department of Mathematics and Statistics & Centre for Artificial Intelligence Banasthali Vidyapith, Banasthali-304022, Rajasthan.
Visit: https://tinyurl.com/SSCA26ConfRegistrationhttps://drive.google.com/file/d/1O9CRTs3OP4N2JM38Hufh2MEw3Vu-yEBh/view?usp=sharing

Name: International conference on Recent Advances in Applied Mathematics (RAAM 2024).
Date: July 03, 2024–July 05, 2024
Venue: Department of Mathematics, IIT (BHU), Varanasi, Uttar Pradesh, India.
Visit: https://conferences.iitbhu.ac.in/RAAM2024/

Name: International Conference on Computations and Data Science.
Date: March 08, 2024–March 10, 2024
Venue: Department of Mathematics, IIT Roorkee.
Visit: https://www.iitr.ac.in/cods24/index.html?

Details of Workshops/Conferences Abroad

Name: Geometry, Statistical Mechanics, And Integrability
Date: March 11, 2024–June 14, 2024
Venue: Institute For Pure And Applied Mathematics (IPAM), Los Angeles, CA, USA.
Visit: `www.ipam.ucla.edu/programs/long-programs/geometry-statistical-mechanics-and-integrability/`

Name: AIM Workshop: Degree D Points On Algebraic Surfaces
Date: March 18, 2024–March 22, 2024
Venue: American Institute Of Mathematics, Pasadena, California, USA.
Visit: `aimath.org/workshops/upcoming/degreedsurface/`

Name: Analysis On Fractals And Networks, And Applications
Date: March 18, 2024–March 22, 2024
Venue: CIRM, 163 Avenue De Luminy, Case 916 13288 Marseille Cedex 9, FRANCE.
Visit: `conferences.cirm-math.fr/2950.html`

Name: Multi-Scale Methods For Reactive Flow And Transport In Complex Elastic Media, Conference In Memory Of Prof. Andro Mikelic
Date: March 19, 2024–March 22, 2024
Venue: CAAC, Center For Advanced Academic Studies, Dubrovnik, Croatia.
Visit: `web.math.pmf.unizg.hr/andromikelic/`

Name: Workshop I: Statistical Mechanics And Discrete Geometry
Date: March 25, 2024–March 29, 2024
Venue: Institute For Pure And Applied Mathematics (IPAM), Los Angeles, CA, USA.
Visit: `www.ipam.ucla.edu/programs/workshops/workshop-i-statistical-mechanics-and-discrete-geometry/`

Name: Modern Aspects Of Harmonic Analysis On Lie Groups
Date: April 2, 2024–April 5, 2024
Venue: Georg-August-University GÖTtingen, GÖTtingen, Lower-Saxony/Germany.
Visit: `jaeh.cc/SS2024/index.htm`

Name: Recent Developments In Noncommutative Algebraic Geometry
Date: April 8, 2024–April 12, 2024
Venue: SLMath 17 Gauss Way, Berkeley, CA 94720, USA.
Visit: `www.msri.org/workshops/1075`

Name: Workshop II: Integrability And Algebraic Combinatorics
Date: April 15, 2024–April 19, 2024
Venue: Institute For Pure And Applied Mathematics (IPAM), Los Angeles, CA, USA.
Visit: `www.ipam.ucla.edu/programs/workshops/workshop-ii-integrability-and-algebraic-combinatorics/`

Name: Recent Developments In Commutative Algebra
Date: April 15, 2024–April 19, 2024
Venue: SLMath 17 Gauss Way, Berkeley, CA 94720, USA.
Visit: `www.msri.org/workshops/1060`

Name: AIM Workshop: Higher-Dimensional Contact Topology
Date: April 15, 2024–April 19, 2024
Venue: American Institute Of Mathematics, Pasadena, California, USA.
Visit: `aimath.org/workshops/upcoming/highdimcontacttop/`

Name: CRM Thematic Semester On "Geometric Analysis"
Date: April 15, 2024–April 29, 2024
Venue: Centre De Recherches Mathématiques, Université De Montréal, Québec, Canada.
Visit: www.crmath.ca/en/activities/{#}/type/activity/id/3880

Name: SIAM Conference On Data Mining (SDM24)
Date: April 18, 2024–April 20, 2024
Venue: Westin Houston, Memorial City, Houston, Texas, USA.
Visit: www.siam.org/conferences/cm/conference/sdm24

Name: International Summit On Materials Science
Date: April 19, 2024–April 20, 2024
Venue: Tokyo, Japan.
Visit: materialsscience.averconferences.com/

Name: AIM Workshop: Post-Quantum Group-Based Cryptography
Date: April 29, 2024–May3, 2024
Venue: American Institute Of Mathematics, Pasadena, California, USA.
Visit: aimath.org/workshops/upcoming/postquantgroup/

Name: Advances In Lie Theory, Representation Theory And Combinatorics: Inspired By The Work Of Georgia M. Benkart
Date: May 1, 2024–May 3, 2024
Venue: SL Math 17 Gauss Way, Berkeley, CA 94720, USA.
Visit: www.msri.org/workshops/1065/

Name: Workshop III: Statistical Mechanics Beyond 2D
Date: May 6, 2024–May 10, 2024
Venue: Institute For Pure And Applied Mathematics (IPAM), Los Angeles, CA, USA.
Visit: www.ipam.ucla.edu/programs/workshops/workshop-iii-statistical-mechanics-beyond-2d/

Name: AIM Workshop: High-Dimensional Phenomena In Discrete Analysis
Date: May 13, 2024–May 17, 2024
Venue: American Institute Of Mathematics, Pasadena, California, USA.
Visit: aimath.org/workshops/upcoming/highdimdiscrete/

Name: SIAM Conference On Applied Linear Algebra (LA24)
Date: May 13, 2024–May 17, 2024
Venue: Sorbonne Universite, Paris, France.
Visit: www.siam.org/conferences/cm/conference/la24

Name: SIAM Conference On Mathematical Aspects Of Material Science (MS24)
Date: May 19, 2024–May 23, 2024
Venue: Sheraton Pittsburgh Station Square, Pittsburgh, Pennsylvania, USA.
Visit: www.siam.org/conferences/cm/conference/ms24

Name: Workshop IV: Vertex Models: Algebraic And Probabilistic Aspects Of Universality
Date: May 20, 2024–May 24, 2024
Venue: Institute For Pure And Applied Mathematics (IPAM), Los Angeles, CA, USA.
Visit: www.ipam.ucla.edu/programs/workshops/workshop-iv-vertex-models-algebrai c-and-probabilistic-aspects-of-universality/

Name: XXII GEOMETRICAL SEMINAR
Date: May 26, 2024–May 31, 2024
Venue: Vrnjaéka Banja, Serbia.
Visit: tesla.pmf.ni.ac.rs/people/geometrijskiseminarxxii/

Name: Representation Theory And Related Geometry: Progress And Prospects (On The Occasion Of Daniel K. Nakano's 60th Birthday)
Date: May 27, 2024–May 31, 2024
Venue: University Of Georgia, Athens, GA, USA.
Visit: sites.google.com/view/representation-theory-geometry

Name: SIAM Conference On Imaging Science (IS24)
Date: May 28, 2024–May 31, 2024
Venue: Westin Peachtree Plaza, Atlanta, Georgia, USA.
Visit: www.siam.org/conferences/cm/conference/is24

Name: Computational Aspects Of Thin Groups
Date: June 3, 2024–June 14, 2024
Venue: IMS, National University Of Singapore.
Visit: ims.nus.edu.sg/events/computational-aspects-of-thin-groups/

Name: Séminaire De MathématiquesSupérieures 2024: "Flows And Variational Methods InRiemannian And Complex Geometry: Classical And Modern Methods (Montréal, Canada)"
Date: June 3, 2024–June 14, 2024
Venue: Montréal, Canada.
Visit: www.slmath.org/summer-schools/1061

Name: BIOMATH 2024: International Conference On Mathematical Methods And Models In Biosciences
Date: June 16, 2024–June 22, 2024
Venue: Cutty Sark Resort, Scottburgh, South Africa.
Visit: biomath.bg/2024

Name: Open Communications In Nonlinear Mathematical Physics - 2024
Date: June 23, 2024–June 29, 2024
Venue: Häcker's Grand Hotel, Bad Ems, Rhineland-Palatinate, Germany.
Visit: euler-ocnmp.de/

Name: New Perspectives In Computational Group Theory
Date: June 24, 2024–June 26, 2024
Venue: University Of Warwick, UK.
Visit: sites.google.com/view/newperspectivescgt/home

Name: SIAM Conference On Nonlinear Waves And Coherent Structures (NWCS24)
Date: June 24, 2024–June 27, 2024
Venue: Lord Baltimore Hotel, Baltimore, MD, USA.
Visit: www.siam.org/conferences/cm/conference/nwcs24

Name: ICERM Workshop: Queer In Computational And Applied Mathematics
Date: June 24, 2024–June 28, 2024
Venue: ICERM (Providence, Rhode Island), USA.
Visit: icerm.brown.edu/topical_workshops/tw-24-qcam/

The Mathematics Newsletter may be downloaded from the RMS website at

www.ramanujanmathsociety.org

MATHEMATICS NEWSLETTER

Mathematics Newsletter is a quarterly journal published in March, June, September and December of each year. The first issue of any new volume is published in June.

Mathematics Newsletter welcomes from its readers

- Expository articles in mathematics typed in LaTeX or Microsoft Word;
- Information on forthcoming meetings, seminars, workshops and conferences in mathematics and reports on those which were recently concluded;
- Mathematical puzzles and problems addressed to the readership of the *Newsletter;*
- Solutions to mathematical problems that have appeared in the *Newsletter* and comments on the solutions;
- Brief reports on the mathematical activities at their departments that might be of interest to the readership of the *Newsletter;*
- Information about faculty positions and scholarships;
- Abstracts (each not exceeding one page) of recent Ph.D. theses;
- Descriptions of recently-published books written by them; and
- Any other items that might be of interest to the mathematical community.

Readers are requested not to submit regular research articles for publication in the *Mathematics Newsletter*. The *Newsletter* is not the forum for such articles. Instead, the *Newsletter* looks for expository articles that are consciously written in a style that would make them accessible to a broad mathematical readership.

- Chief Editor, Mathematics Newsletter

MATHEMATICS NEWSLETTER

Volume 34 December 2023 No. 3

CONTENTS

The generalized Eulerian power sums $\sum_{k=1}^{n} k^m z^k$ ... Raymond Mortini 29

A review on the closed graph theorem and the open mapping theorem via finite dimensionality ... Uday Shankar Chakraborty 33

The Andersén-Lempert theory on $\mathbb{C}^n$ and some of its applications ... Sanjoy Chatterjee and Sushil Gorai 35

Quasiconformal extensions of conformal mappings ... Bappaditya Bhowmik 47

Twenty nineth ICFIDCAA-2023 conference report ... 53

Report on the International Conference on Mathematics and Computing-ICMC (January 04–07, 2024) and a Pre-Conference National Symposium on "Advanced Mathematical Methods" (January 02–03, 2024) ... 54

Report on ICAADM 2024 ... 55

Report on Workshop on Algebra, Analysis and Number Theory-2024 ... 55

International Workshop on Geometric Function Theory (IWGFT 2023) ... 55

Report on Teachers' Enrichment Workshop on Linear Algebra, Real Analysis and Topology ... 56

Report on 38th Annual Conference of the Ramanujan Mathematical Society ... 56

Announcement of IITH Institute Postdoctoral Fellowship (IITH-IPDF) (Call is Open now) ... 57

Details of Workshop/Conferences in India ... 57

Details of Workshop/Conferences Abroad ... 58

Visit: www.ramanujanmathsociety.org

Typeset in LaTeX at Krishtel eMaging Solutions Pvt. Ltd., Chennai - 600 087. Phone: 2486 13 16 and printed at United Bind Graphics, Chennai - 600 010. Phone: 9282102533/79692738