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On Some Isometries on Certain Function Spaces

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Abstract. Let $\mathcal{X}$ be the space $C[a,b]$ or $L^\infty[a,b]$ with norm $\|\cdot\|_\infty$ and let $\mathcal{Y}$ be the space $L^1[a,b]$ or $C[a,b]$ with norm $\|\cdot\|_1$. By a simple procedure, we obtain linear isometries from the space $\mathcal{Y}$ to the dual of $\mathcal{X}$. We also show that $L^1[a,b]$ is linearly isometric with the hyperspace $AC_0[a,b] := \{x \in AC[a,b],\ x(a) = 0\}$ of the space $AC[a,b]$, the space of all absolutely continuous functions with total variation as the norm.

Keywords. Isometry, Integrable functions, Essentially bounded functions, Absolutely continuous functions, Functions of bounded variation, Hyperspace.

AMS Subject Classification: 28A25, 28C05, 46B04, 46B10.

1. Introduction

Let $\mathcal{X}$ be the space $C[a,b]$ or $L^\infty[a,b]$ with norm $\|\cdot\|_\infty$ and let $\mathcal{Y}$ be the space $L^1[a,b]$ or $C[a,b]$ with norm $\|\cdot\|_1$. The scalar field for all spaces under consideration is $\mathbb{K}$, the field $\mathbb{R}$ or $\mathbb{C}$. From basic functional analysis (see, e.g., [1],[2]), we may recall the following:

- $C[a,b]$ is the vector space of all continuous $\mathbb{K}$-valued functions on $[a,b]$, and $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms on $C[a,b]$ defined by
$$\|x\|_1 = \int_a^b |x(t)|dt, \quad \|x\|_\infty = \max_{a\le t\le b} |x(t)|,$$
respectively, for $x \in C[a,b]$. With respect to the norm $\|\cdot\|_\infty$, $C[a,b]$ is a Banach space.
- $L^1[a,b]$ is the vector space of all Lebesgue integrable $\mathbb{K}$-valued functions on $[a,b]$ with equality of two elements in it understood in the sense of equal almost everywhere, and $L^1[a,b]$ is a Banach space with respect to the norm
$$\|x\|_1 = \int_a^b |x(t)|dt, \quad x \in L^1[a,b],$$
where the integration is with respect to the Lebesgue measure on $[a,b]$.
- $C[a,b]$ is a dense subspace of $L^1[a,b]$.
- $L^\infty[a,b]$ is the vector space of all $\mathbb{K}$-valued measurable functions on $[a,b]$ which are *essentially bounded*, in the sense that, a measurable function $x : [a,b] \to \mathbb{K}$ belongs to $L^\infty[a,b]$ if and only if there exists $M_x > 0$ such $|x(t)| \le M_x$ for *almost all* (a.a.) $t \in [a,b]$. It is a Banach space with respect to the norm
$$\|x\|_\infty := \inf\{M_x : |x(t)| \le M_x \text{ for a.a. } t \in [a,b]\},$$
which is called the *essential bound* of x.
- $C[a,b]$ is a closed proper subspace of $L^\infty[a,b]$.
- The space of all continuous linear functionals on a normed linear space X, called dual of X and denoted by X', is a Banach space with respect to the norm
$$\|f\| := \sup\{|f(x)| : x \in X,\ \|x\| \le 1\}, \quad f \in X'.$$

For $y \in \mathcal{Y}$, let $f_y : \mathcal{X} \to \mathbb{K}$ be defined by
$$f_y(x) = \int_a^b x(t)y(t)\,dt, \quad x \in \mathcal{X}, \tag{1.1}$$
where the integration is with respect to the Lebesgue measure on $[a,b]$. Then we see that f_y is a linear functional on $\mathcal{X}$ and
$$|f_y(x)| \le \|y\|_1\|x\|_\infty \quad \forall\, x \in \mathcal{X}.$$
Thus, for every $y \in \mathcal{Y}$, f_y is continuous so that it belongs to $\mathcal{X}'$, the dual of $\mathcal{X}$, and
$$\|f_y\| \le \|y\|_1 \quad \forall\, y \in \mathcal{Y} \tag{1.2}$$

so that the map $T : \mathcal{Y} \to \mathcal{X}'$ defined by

$$Ty = f_y, \quad y \in \mathcal{Y}, \tag{1.3}$$

is a bounded linear operator.

In the next section, we show that T is a linear isometry, that is, $\|f_y\| = \|y\|_1$ for all $y \in \mathcal{Y}$, under various choices of $\mathcal{X}$ and $\mathcal{Y}$. We also show that $L^1[a,b]$ is linearly isometric with the hyperspace $AC_0[a,b] := \{x \in AC[a,b], x(a) = 0\}$ of the space $AC[a,b]$, the space of all absolutely continuous functions with total variation as the norm.

2. Main Results

Let $y \in \mathcal{Y}$ and let $\varepsilon > 0$ be given. Then we observe that

$$\begin{aligned}\int_a^b (|y(t) - \varepsilon)dt &\le \int_a^b \frac{|y(t)|^2}{|y(t)| + \varepsilon}dt \\ &= \int_a^b \frac{\overline{y(t)}}{|y(t)| + \varepsilon} y(t)dt \\ &= \int_a^b x_\varepsilon(t) y(t)\, dt,\end{aligned}$$

where

$$x_\varepsilon(t) = \frac{\overline{y(t)}}{|y(t)| + \varepsilon}.$$

Note that

(1) $y \in L^1[a,b]$ implies $x_\varepsilon \in L^\infty[a,b]$,
(2) $y \in C[a,b]$ implies $x_\varepsilon \in C[a,b] \subseteq L^\infty[a,b]$

with $\|x_\varepsilon\|_\infty \le 1$. Thus, for the following choices

(1) $\mathcal{X} = L^\infty[a,b]$ and $\mathcal{Y} = L^1[a,b]$,
(2) $\mathcal{X} = C[a,b]$ with $\|\cdot\|_\infty$ and $\mathcal{Y} = C[a,b]$ with $\|\cdot\|_1$,
(3) $\mathcal{X} = L^\infty[a,b]$ and $\mathcal{Y} = C[a,b]$ with $\|\cdot\|_1$,

we have

$$\int_a^b (|y(t) - \varepsilon)dt \le \int_a^b x_\varepsilon(t) y(t)\, dt = f_y(x_\varepsilon)$$

for $y \in \mathcal{Y}$ and $\varepsilon > 0$. Consequently, for the above choices of the ordered pair $(\mathcal{X}, \mathcal{Y})$, we have

$$\|y\|_1 \le \|f_y\|.$$

Thus, in view of (1.2), we have proved the following theorem.

Theorem 2.1. *Let the ordered pair $(\mathcal{X}, \mathcal{Y})$ be as in any of the following three choices:*

(1) $\mathcal{X} = L^\infty[a,b]$ and $\mathcal{Y} = L^1[a,b]$,
(2) $\mathcal{X} = C[a,b]$ with $\|\cdot\|_\infty$ and $\mathcal{Y} = C[a,b]$ with $\|\cdot\|_1$,
(3) $\mathcal{X} = L^\infty[a,b]$ and $\mathcal{Y} = C[a,b]$ with $\|\cdot\|_1$.

Then the map $T : \mathcal{Y} \to \mathcal{X}'$ defined by (1.3) is a linear isometry.

Remark 2.2. It is a well-known result that the map $T : L^1[a,b] \to (L^\infty[a,b])'$ defined by (1.3) is a linear isometry (see e.g. [1],[2]). We provided above a new proof for this result which helped us unifying this result with other choices of ordered pair $(\mathcal{X}, \mathcal{Y})$ as in (2) in the above theorem. It is also to be mentioned here that Theorem 2.1 can be easily extended to the case wherein the interval $[a,b]$ is replaced by the closure of any open bounded subset of $\mathbb{R}^d$ for any $d \in \mathbb{N}$.

From Theorem 2.1, we derive the following.

Theorem 2.3. *Let $\mathcal{X} = (C[a,b], \|\cdot\|_\infty)$ and $\mathcal{Y} = L^1[a,b]$. Then the map $T : \mathcal{Y} \to \mathcal{X}'$ defined by (1.3) is a linear isometry.*

Proof. By Theorem 2.1, the map $y \mapsto f_y$ from $(C[a,b], \|\cdot\|_1)$ to $(C[a,b], \|\cdot\|_\infty)'$ is a linear isometry. Now, the conclusion in the theorem follows using the fact that $(C[a,b], \|\cdot\|_1)$ is dense in $L^1[a,b]$.

For the next result, let us consider the space

$$AC_0[a,b] := \{x \in AC[a,b], x(a) = 0\}$$

with total variation as the norm, where $AC[a,b]$ is the space of all absolutely continuous functions on $[a,b]$. On $AC_0[a,b]$ we consider the norm as the total variation, and show that the map $y \mapsto f_y$ is a linear isometry from $L^1[a,b]$ *onto* $AC_0[a,b]$. Before this, let us define certain notions involved in the above.

A function $\varphi : [a,b] \to \mathbb{K}$ is said to be *absolutely continuous* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every disjoint family of intervals $(s_i, t_i) \subseteq [a,b]$, $i = 1, \ldots, n$,

$$\sum_{i=1}^n (t_i - s_i) < \delta \implies \sum_{i=1}^n |\varphi(t_i) - \varphi(s_i)| < \varepsilon.$$

It is known that every absolutely continuous function is of *bounded variation* (cf. [3]). We may recall that (cf. [1],[2],[3]) a function $\varphi : [a,b] \to \mathbb{K}$ is said to be of *bounded* *variation* if its *total variation* $V(\varphi)$ is finite, where

$$V(\varphi) := \sup_{\mathcal{P}} \sum_{i=1}^{k} |\varphi(t_i) - \varphi(t_{i-1})| < \infty,$$

where the supremum is taken over all partitions $\mathcal{P} : a := t_0 < t_1 < \cdots < t_k$ of $[a, b]$. It is also known that

$$\|\varphi\|_{BV} := |\varphi(a)| + V(\varphi)$$

is a norm on $BV[a, b]$, the vector space of all functions of bounded variation.

By *fundamental theorem of Lebesgue integration* (cf. [3]), if $y \in L^1[a, b]$, then the function $v_y : [a, b] \to \mathbb{K}$ defined by

$$v_y(t) = \int_a^t y(s)\, ds, \quad t \in [a, b], \tag{2.1}$$

belongs to $AC_0[a, b]$, v_y differentiable almost everywhere and $v_y' = y$ a.e., and conversely, if $v \in AC_0[a, b]$ then it is differentiable a.e., $v' \in L^1[a, b]$ and

$$v(t) = \int_a^t v'(s)\, ds, \quad t \in [a, b].$$

Now, let $y \in L^1[a, b]$ and v_y be as in (2.1). Then we have $v_y \in AC_0[a, b]$. In particular, $v_y \in BV[a, b]$ with norm $\|v_y\|_{BV} = V(v_y)$. In fact,

$$\|v_y\|_{BV} \leq \|y\|_1.$$

To see this, consider a partition $\mathcal{P} : a = t_0 < t_1 < \cdots < t_k = b$ of $[a, b]$. Then we have

$$\begin{aligned}\sum_{i=1}^{k} |v_y(t_i) - v_y(t_{i-1})| &= \sum_{i=1}^{k} \left| \int_{t_{i-1}}^{t_i} y(s) ds \right| \\ &\leq \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} |y(s)| ds \\ &= \int_a^b |y(s)| ds = \|y\|_1.\end{aligned}$$

Hence, $\|v_y\|_{BV} = V(y) \leq \|y\|_1$.

Theorem 2.4. *Let $y \in L^1[a, b]$ and $v_y : [a, b] \to \mathbb{K}$ be defined by*

$$v_y(t) = \int_a^t y(s)\, ds, \quad t \in [a, b].$$

Then $v_y \in AC_0[a, b]$ and the map $y \mapsto v_y$ is a linear isometry from $L^1[a, b]$ onto $AC_0[a, b]$ with the norm $\|\cdot\|_{BV}$ on $AC_0[a, b]$.

Proof. Let $y \in L^1[a, b]$. We have seen that $v_y \in AC_0[a, b]$ and $\|v_y\|_{BV} \leq \|y\|_1$. Also, by fundamental theorem of Lebesgue integration, the map $y \mapsto v_y$ from $L^1[a, b]$ to $AC_0[a, b]$ is onto. Hence, it is enough to show that $\|y\|_1 \leq \|v_y\|_{BV}$.

By Theorem 2.3, we know that

$$\|y\|_1 = \|f_y\|,$$

where $f_y \in (C[a, b], \|\cdot\|_\infty)'$ defined by (1.1), that is,

$$f_y(x) = \int_a^b x(t)y(t)\, dt, \quad x \in C[a, b].$$

Since $y = v_y'$ a.e., we have

$$\int_a^b x(t)y(t)\, dt = \int_a^b x(t)v_y'(t)\, dt = \int_a^b x(t)dv_y(t)\, dt,$$

where the last integral is in the sense of Riemann-Stiltjes. Hence,

$$|f_y(x)| \leq \|x\|_\infty \|v_y\|_{BV} \quad \forall x \in C[a, b].$$

Hence, $\|f_y\| \leq \|v_y\|_{BV}$ so that we have

$$\|y\|_1 = \|f_y\| \leq \|v_y\|_{BV}.$$

This completes the proof.

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Fatou-Bieberbach domains and attracting basins

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Abstract. Fatou-Biberbach domains in $\mathbb{C}^k, k \geq 2$ are proper subdomains of $\mathbb{C}^k, k \geq 2$ that are biholomorphic to $\mathbb{C}^k, k \geq 2$. Such domains clearly do not exist in $\mathbb{C}$ and, further arise naturally from dynamics of automorphisms of $\mathbb{C}^k, k \geq 2$, admitting an attracting fixed point. The goal of this article is to discuss a few pathological examples of such domains, and also their relevance in the context of geometry, complex function theory and dynamics.

Furthermore we discuss non-autonomous (dynamical) families and their basins attractions, and briefly outline an (affirmative) solution to a long standing open problem, called the Bedford's conjecture, related to the (non-autonomous) dynamical origins of a Fatou-Bieberbach domain.

Keywords. Hénon maps, Semigroups, Julia and Fatou sets, Fatou-Bieberbach domains.

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1. Introduction

The goal of this paper is to provide a brief survey on Fatou-Bieberbach domains in $\mathbb{C}^k$, $k \geq 2$.

Definition. *A proper subdomain Ω of $\mathbb{C}^k$, $k \geq 2$ is said to a Fatou-Bieberbach domain (or F.B. domain), if it is biholomorphic to $\mathbb{C}^k$.*

In particular, there exist holomorphic injective maps $\phi : \mathbb{C}^k \to \mathbb{C}^k$, which are not onto. The domain $\phi(\mathbb{C}^k) \subsetneq \mathbb{C}^k$, gives a Fatou-Bieberbach domain. It may be noted that as a consequence of the Picard's theorem, which states that image of an entire function $\phi : \mathbb{C} \to \mathbb{C}$ is either whole $\mathbb{C}$ or $\mathbb{C}$ minus a point, such domains do not exist in $\mathbb{C}$.

The goal of this write-up is to discuss Fatou-Bieberbach domains and their pathologies from the context of geometry, function theory and dynamics. The existence of Fatou-Bieberbach domain was confirmed/predicted by the French mathematician Fatou in 1922 in [9] and later an explicit construction was given by Bieberbach in 1930 in [8], using a method that dates back to Poincaré (around 1890). Bieberbach's method involved the following three basic steps

- Constructing a sequence of automorphism $\{\phi_n\}$ of $\mathbb{C}^k$, $k \geq 2$ such that Jacobian of $\phi_n \equiv 1$ for every $n \geq 1$.
- The limit of ϕ_n exists over all compact subsets K of $\mathbb{C}^k$ such that the Jacobian of the limit map ϕ is also 1.
- The limit map $\phi : \mathbb{C}^k \to \mathbb{C}^k$ is injective by construction, but the range of ϕ is not dense in $\mathbb{C}^k$. Hence $\Omega = \phi(\mathbb{C}^k)$ is a Fatou-Bieberbach domain.

Apparently one might feel that Fatou-Bieberbach domains are large, in particular their measure is infinite. But Fatou-Bieberbach domains with pathological properties, i.e., both small and big Fatou-Bieberbach domains have been constructed by various authors. A few important constructions and an open question on Fatou-Bieberbach domains are enlisted as follows

(a) **Small F.B. domains:** (Rosay-Rudin [18]) There exist a sequence of F.B. domains (say Ω_n) in $\mathbb{C}^k$, $k \geq 2$ such that

$$\Omega_n \cap \Omega_m = \emptyset, m \neq n \text{ and } \bigcup_{n=1}^{\infty} \Omega_n \neq \mathbb{C}^k.$$

(b) **Large F.B. domains:** (Wold [20]) For a given (polynomially) convex compact set $K \subset \mathbb{C}^k$ $(k \geq 2)$ and a countable dense subset E of $\mathbb{C}^k \setminus K$, there exist a F.B. domain avoiding K but containing E.

(c) **F.B. domains with chaotic boundary:** (Peters-Wold [17]) There exist F.B. domain in $\mathbb{C}^2$, with boundary having *Hausdorff dimension* 4.

(d) **F.B. domains with smooth boundary:** (Stensønes [19]) There exist F.B. domains in $\mathbb{C}^k, k \geq 2$, with smooth boundary.

(e) **Open Question:** Is there an F.B. domain with real analytic boundary, i.e., whether the boundary of the Fatou-Bieberbach domain is realised as a zero set of a (locally) real-analytic function or not?

A general method to construct Fatou-Bieberbach domains was developed by Rosay-Rudin in 1989 (see [18]), obtained as a consequence of dynamical properties of automorphisms of $\mathbb{C}^k$, $k \geq 2$. Their result is stated as

Theorem (Rosay-Rudin, 1989). *If $F \in \mathrm{Aut}(\mathbb{C}^k)$ with an attracting fixed point at $p \in \mathbb{C}^k$ then the basin of attraction of F at p, i.e.,*

$$\Omega_p^F := \{z \in \mathbb{C}^k : F^n(z) = \overbrace{F \circ F \circ F \circ \cdots \circ F}^{n\text{-times}}(z) \to p \text{ as } n \to \infty\} \simeq \mathbb{C}^k.$$

Example. As a consequence of the above theorem one obtains that if H is a Hénon map of the form

$$H(x, y) = (y, ax + y^2), 0 < |a| < \frac{1}{2},$$

then $\Omega^H_{(0,0)} \simeq \mathbb{C}^2$. But now it is easy to check that the point $(0, 3) \notin \Omega^H_{(0,0)}$, hence $\Omega^H_{(0,0)}$ is an F.B. domain.

In Section 2, we discuss the connection of Fatou-Bieberbach domains to function theory in several variables. In this context we will emphasize another class of domains called the *Short* $\mathbb{C}^2$*'s*. The construction of Short $\mathbb{C}^2$'s was given by Fornæss in [10], and the method is similar to Rosay-Rudin's construction of F.B. domains in spirit. Thus in this context, we also note a dictionary of pathological Short $\mathbb{C}^k$'s, at par with F.B. domains.

In Section 3, we discuss a long-standing open problem in holomorphic dynamics, raised by Bedford in 2000, on uniformisation of stable manifolds. Due to certain equivalent formulations this question has a deep connection to Fatou-Bieberbach domains. In particular, it boils down to a question generalising Rosay-Rudin's formula on existence of Fatou-Bieberbach domains (see [12]). In the rest of this section we will briefly discuss the main ideas of our work [7], which gives an affirmative answer to Bedford's question.

2. Fatou-Bieberbach domains and the union problem

As mentioned in the introduction, here we discuss the importance of Fatou-Bieberbach domains from the point of view of function theory, in particular in the context of the classical Levi problem. For this we first recall a definition: A domain $\Omega \in \mathbb{C}^k$ is said to be a *domain of holomorphy* if it admits a holomorphic function that does not extend analytically beyond any boundary point of the domain Ω.

Theorem 2.1 (E.E. Levi, 1910, [14]). *Every (smooth) domain of holomorphy is a pseudoconvex domain, i.e.,* Ω *admits a continuous plurisubharmonic exhaustion function.*

Further he raised the converse question (in [14]), which is known in the literature as the

Problem (Levi Problem). *Whether every pseudoconvex domain in* $\mathbb{C}^k$, $k \geq 2$, *is a domain of holomorphy?*

The Levi problem was solved affirmatively first by Oka ([16]) and then by Hörmander ([13]) in 1930's. But an important contribution in the study of Levi Problem — in particular a result that helped in removing the assumption of smooth boundary conditions from the Levi Problem — is

Theorem (Behnke-Stein, 1938, [3]). *Increasing or ascending union of domain of holomorphy is also a domain of holomorphy.*

Note here that F.B. domains are also obtained as limits of increasing union of balls, and biholomorphic images of balls are domains of holomorphy. Thus as a consequence of the Behnke-Stein theorem, an F.B. domain is a domain of holomorphy or equivalently, due to the Levi problem, is also a pseudoconvex domain. This resulted in posing the natural question, popularly termed as the union problem.

Problem (Union Problem for balls). *Is it possible to classify domains that are obtained as increasing or ascending limits of domains biholomorphic to the balls in* $\mathbb{C}^k$, $k \geq 2$?

A concrete answer was given to the union problem in $\mathbb{C}^2$ by Fornæss and Sibony in 1981 ([11]) stated as

Theorem 2.2. *Let Ω be a subdomain of $\mathbb{C}^2$ obtained as increasing union of biholomorphic images of the ball such that the* **infinitesimal Kobayashi metric does not vanish identically**. *Then Ω is biholomorphic to the unit ball in $\mathbb{C}^2$ or $\mathbb{D} \times \mathbb{C}$.*

To mention here, we say that the *infinitesimal Kobayashi metric on a domain $\Omega \subset \mathbb{C}^k$ vanishes identically* if for every point $p \in \Omega$ and any non-zero vector $\xi \in \mathbb{C}^k$ (however arbitrarily large in magnitude) there exists a holomorphic function $\rho_{p,\xi} : \mathbb{D} \to \Omega$ such that

$$\rho_{p,\xi}(0) = p \text{ and } D\rho_{p,\xi}(0) = \xi.$$

Geometrically the above signifies that it can fit in arbitrarily large discs as image of holomorphic functions at every point p in the domain Ω. Further the above property is invariant under biholomorphisms.

Now note that it is easy to observe that in the whole of $\mathbb{C}^k$ the Kobayashi metric vanishes identically—consider the function

$$\rho_{p,\xi}(z) = p + z\xi \text{ for every } p \in \mathbb{C}^k, z \in \mathbb{D}.$$

Thus the Kobayashi metric vanishes identically on Fatou-Bieberbach domains as well. Thus it poses the following question

Problem. *What is the analogue of Fornæss-Sibony's result when the Kobayashi metric vanishes identically on the domain Ω.*

In this context, there exists the class of domains, called **Short** $\mathbb{C}^k$'s—obtained by a similar (non-autonomous) dynamical process as Rosay-Rudin's by Fornæss in 2004 (see [10]).

Theorem 2.3 (Fornæss, 2004). *Let $F_n(z, w) = (a_n w + z^2, a_n z)$ where $0 < |a_{n+1}| < |a_n|^2$ and $0 < |a_1| < 1$. Then the basin of attraction of F_n's at the origin, i.e.,*

$$\Omega_{F_n} := \{z \in \mathbb{C}^2 : F(n)(z) = F_n \circ F_{n-1} \circ \cdots \circ F_1(z) \to 0 \text{ as } n \to \infty\}$$

is not a Fatou–Bieberbach domain. The properties of Ω_{F_n} are as follows

(i) Ω_{F_n} can be written as increasing union of domains, biholomorphic to the unit ball.

(ii) The infinitesimal Kobayashi metric vanishes identically on Ω_{F_n}.

(iii) Ω_{F_n} admits a non–constant bounded plurisubharmonic function.

Here we mention the following result (see [6] for details), which essentially nullifies the hope of a concrete classification result for the union problem (for balls) with identically vanishing infinitesimal Kobayashi metric in the limit domain. It is stated as

Theorem (Bera-Pal-Verma, 2018). *There exists a continuum of biholomorphic non-equivalent Short $\mathbb{C}^2$'s.*

As mentioned before note that the existence of short $\mathbb{C}^k$'s is motivated from the existence of Fatou-Bieberbach domains by Rosay-Rudin. Hence it is natural to ask—whether we have pathological examples of short $\mathbb{C}^k$'s as well. So we mention the following series of results and an open problem, obtained keeping in mind the analogue dictionary for F.B. domains.

(a) *(Bera, 2018, [4])* There exists a disjoint union of collection of short $\mathbb{C}^2$'s in $\mathbb{C}^2$.

(b) *(Bera, 2018, [4])* Given a polynomially convex set K there exists a short $\mathbb{C}^2$ which is dense in the complement of K.

(c) *(Fornaess, 2004, [10])* There exists a short $\mathbb{C}^2$ that properly contains an F.B. domain and has smooth boundary.

(d) *(Bera, 2018, [4])* For every $s \in [3, 4]$ there exists a short $\mathbb{C}^2$ with a boundary having Hausdorff dimension s.

(e) *Open question:* Is there a short $\mathbb{C}^2$ with real analytic boundary?

3. Fatou-Bieberbach domains and the stable manifolds

In this section, we discuss the natural connection between Fatou-Bieberbach domains and the stable manifolds. It is known that Rosay-Rudin's result on attracting basins can be equivalently stated in the context of (attracting) stable manifolds as

Theorem (Sternberg). *Let (M, ρ) be a complex manifold with a Riemannian metric ρ. If F is an automorphism of M*

with a hyperbolic fixed point, i.e., the eigenvalues of F lie in the complement of the unit circle. Then what happens? Incomplete statement. Further, if the stable dimension at p is $k \geq 1$, then the **stable set/manifold** *of F at p, defined as*

$$W_s^F(p) := \{z \in M : F^n(z) = \overbrace{F \circ F \circ F \circ \cdots \circ F}^{n\text{-times}}(z) \to p \text{ as } n \to \infty\}$$

is biholomorphic to $\mathbb{C}^k$.

As a generalisation to the above setup, Bedford raised the following problem in 2000 at [2].

Problem (Bedford, 2000). *Let (M, ρ) be a complex manifold with a Riemannian metric ρ. If F is an automorphism of M such that the action of F is* **uniformly hyperbolic** *on compact* **invariant** *set K with stable dimension $k \geq 1$ for every point $p \in K$, i.e., $\sharp E_s^F(p) = k$ for every $p \in K$, whether for every $p \in K$ the* **stable set/manifold** *of F at p,*

$$W_s^F(p) := \{z \in M : F^n(z) = \overbrace{F \circ F \circ F \circ \cdots \circ F}^{n\text{-times}}(z) \to F^n(p) \text{ as } n \to \infty\}$$

is biholomorphic to $\mathbb{C}^k$? In particular, is every stable manifold corresponding to a hyperbolic action uniformized by $\mathbb{C}^k$?

The above problem was realized as the following conjecture in the context of Fatou-Bieberbach domains (or in the context of a generalisation to Rosay-Rudin's result) by Fornæss-Stensønes in 2004 (see [12]). The same is also popularized as Bedford's conjecture and is stated as

Conjecture. (Fornæss-Stensønes, 2004 or Bedford's Conjecture). *Suppose $\{F_n\}$ is a sequence of automorphisms of $\mathbb{C}^k$ with attracting fixed point at the origin and there exist $0 < A < B < 1$, such that*

$$A\|z\| \leq \|F_n(z)\| \leq B\|z\| \text{ for every } z \in B(0; r).$$

Then the **non-autonomous basin** *of attraction of the sequence $\{F_n\}$ at the origin, defined as*

$$\Omega_0^{\{F_n\}} := \{z \in \mathbb{C}^k : F(n)(z) = F_n \circ F_{n-1} \circ \cdots \circ F_1(z) \to 0 \text{ as } n \to \infty\} \simeq \mathbb{C}^k.$$

The above, i.e., both Bedford's problem and conjecture were open for some time since 2000, with partial answers obtained by various authors. To mention them briefly, in the context of Bedford's open problem an almost affirmative answer was obtained in [15].

Theorem 3.1 (Jonsson-Varolin, 2002). *There exists a probability measure μ on the hyperbolic set K—where K is as in the statement of (Bedford's) problem—such that for almost every point p in the set K, the stable manifold through p is biholomorphic to $\mathbb{C}^k$.*

Later an affirmative answer to Bedford's conjecture—as stated by Fornæss-Stensønes—in the case $B^2 < A$ was obtained in [17]. It is stated as

Theorem 3.2 (Peters-Wold, 2007). *Suppose $\{F_n\}$ is a sequence of automorphism of $\mathbb{C}^k$ with attracting fixed point at the origin and there exist $0 < A < B < 1$ and $B^2 < A$ such that*

$$A\|z\| \leq \|F_n(z)\| \leq B\|z\| \text{ for every } z \in B(0; r). \tag{3.1}$$

Then the non-autonomous basin of attraction of the sequence $\{F_n\}$ at the origin, defined as

$$\Omega_0^{\{F_n\}} := \{z \in \mathbb{C}^k : F(n)(z) = F_n \circ F_{n-1} \circ \cdots \circ F_1(z) \to 0 \text{ as } n \to \infty\} \simeq \mathbb{C}^k.$$

There were further improvements to the above result by various authors, with additional assumption on the behaviour of the sequence of functions, however the statement in its full generality was open, until 2022. In [7], we give an affirmative answer to Bedford's conjecture, thus also answering affirmatively Bedford's (open) problem on uniformisation of stable manifolds. To state the theorem we first note the following definitio: A sequence of automorphisms $\{F_n\}$ of $\mathbb{C}^k$ is said to be **uniformly attracting** at the origin if it satisfies the above condition (3.1) on a uniform neighbourhood of the origin.

Theorem 3.3 (Bera-Verma, 2022). *Suppose $\{F_n\} \in \mathrm{Aut}(\mathbb{C}^k)$ is uniformly attracting at the origin. Then the non-autonomous basin of attraction of the sequence $\{F_n\}$ at the origin, i.e., $\Omega_0^{\{F_n\}}$ is biholomorphic to $\mathbb{C}^k$.*

The proof in $\mathbb{C}^3$ and higher dimensions involves a careful study of the potential theoretic properties of a family of (weak) shift-like maps. The method indeed involves a few technical steps. To continue our discussion here we will briefly discuss the motivation and the idea of the proof of the above theorem in $\mathbb{C}^2$. We first observe that given a uniformly attracting non-autonomous sequence $\{F_n\}$ such that $B^{k_0} < A < B$, by simultaneously triangularizaing the linear part of each $\{F_n\}$ we may assume

$$F_n(x, y) = (a_n x, b_n y + c_n x) + \text{higher order terms}$$

As a Step 1 we first prove the following result in $\mathbb{C}^2$.

Theorem 3.4 (Bera-Verma, 2022). *There exist sequences of polynomials $\{p_n\}$ and $\{q_n\}$ of degree k_0 in one variable with*

$$p_n(0) = q_n(0) = p_n'(0) = 0 \text{ and } q_n'(0) = b_n$$

for every $n \geq 1$ such that the basin of attraction $\Omega_0^{\{g_n\}}$ of the sequence of endomorphisms

$$g_n(x, y) = \Big(a_n x + p_n(y + c_n^{-1} q_n(x)), c_n y + q_n(x)\Big) \quad (3.2)$$

is biholomorphic to $\Omega_0^{\{F_n\}}$.

The above result relies on the idea of **non-autonomous conjugation** of sequences of functions and follows from an important result of Abate-Abbondandolo-Majer (2015), obtained in [1]. Note that the sequence $\{g_n\}$ thus obtained is a uniformly attracting sequence of Hénon maps with degree at most $k_0 \geq 2$. A further analysis of the sequence of Hénon maps obtained above gives that they satisfy the **uniform filtration and uniform bound property.** To state the property precisely—for $R > 0$, consider the filtration of $\mathbb{C}^2$ given by

$$V_R = \{|x|, |y| < R\}, V_R^+ = \{|y| \geq \max\{|x|, R\}\},$$
$$V_R^- = \{|x| \geq \max\{|y|, R\}\}.$$

A sequence of generalized Hénon maps $\{\mathsf{H}_n\}$, (that is, H_n is a finite composition of maps of form

$$\mathsf{H}(x, y) = (y, \delta x + P(y)) \quad (3.3)$$

where $\delta \neq 0$ and P is a polynomial in one variable of degree at least 2), is said to satisfy the *uniform filtration and bound condition* if:

(i) $\{\mathsf{H}_n\}$ admits a uniform filtration radius $R_{\{\mathsf{H}_n\}} > 1$ (sufficiently large) such that for every $R > R_{\{\mathsf{H}_n\}}$

(a) $\overline{\mathsf{H}_n(V_R^+)} \subset V_R^+$ and $\overline{\mathsf{H}_n^{-1}(V_R^-)} \subset V_R^-$,

(b) there exists a sequence of positive real numbers $\{R_n\}$ diverging to infinity, with $R_0 = R$, satisfying $V_{R_n} \cap \mathsf{H}(n)(V_R^+) = \emptyset$ and $V_{R_n} \cap \mathsf{H}(n)^{-1}(V_R^-) = \emptyset$,

(c) there exist uniform positive constants $0 < m < 1 < M$ such that

$$m|y|^{d_n} < \|\mathsf{H}_n(x, y)\|$$
$$= |\pi_2 \circ \mathsf{H}_n(x, y)| < M|y|^{d_n} \text{ on } V_R^+$$

and

$$m|x|^{d_n} < \left\|\mathsf{H}_n^{-1}(x, y)\right\|$$
$$= |\pi_1 \circ \mathsf{H}_n^{-1}(x, y)| < M|x|^{d_n} \text{ on } V_R^-.$$

where d_n is the degree of H_n.

(ii) For every $R \geq R_{\{\mathsf{H}_n\}}$, there exists a uniform constant $B_R = \max\{\|\mathsf{H}_n(z)\| : z \in V_R\} < \infty$.

Thus the proof follows from the following Step 2 which is a Corollary obtained in [5].

Corollary 3.5 (Bera, 2022). *Let $\{h_n\}$ be a sequence of Hénon maps which satisfy the uniform filtration and bound condition and is uniformly attracting on a neighbourhood of origin, i.e., satisfying (3.1). Then the basin of attraction of the sequence $\{h_n\}$ at the origin is biholomorphic to $\mathbb{C}^2$.*

This completes the discussion of the proof in $\mathbb{C}^2$.

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Quasiconformal Mappings and Geometric Function Theory

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Abstract. This is an introductory survey article on quasiconformal mappings and the Geometric Function Theory (GFT for short) related to quasiconformal mappings. We begin with the definition of plane quasiconformal mappings and their basic properties. Then we introduce to various classes of plane domains which appear frequently in GFT with a special emphasis on the relationship with quasiconformal mappings. In the final part, we focus on the coefficient problems for univalent analytic functions on the inside or outside of the unit circle which admit quasiconformal extension to the whole plane or the Riemann sphere.

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1. Plane quasiconformal mappings

Let K be a number with $1 \leq K < +\infty$ and set $k = (K-1)/(K+1) \in [0,1)$. An orientation preserving homeomorphism f of a domain D in the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ onto another domain D' in $\widehat{\mathbb{C}}$ is called a *K-quasiconformal mapping* if f has a locally square integrable derivative (in the sense of distributions) on $D \backslash \{\infty, f^{-1}(\infty)\}$ which satisfy the inequality

$$|f_{\bar{z}}| \leq k|f_z|$$

almost everywhere on D. Here,

$$f_z = \frac{1}{2}(f_x - i f_y) \quad \text{and} \quad f_{\bar{z}} = \frac{1}{2}(f_x + i f_y).$$

Note that f is also called k-quasiconformal and in the large part of the present exposition, we indeed use this terminology. If we do not specify the constant K, the map is simply called quasiconformal. In the above definition, if we replace "orientation preserving homeomorphism" by "continuous map", then the function f is called *K-quasiregular*. In that case, we do not specify the range and say that $f : D \to \widehat{\mathbb{C}}$ is (K-)quasiregular. Note that quasiconformal mappings are quasiregular. A more detailed account is given in [Sug18]. Quasiconformal mappings enjoy the following properties.

Theorem 1.6. *Let D, D', D'' be subdomains of $\widehat{\mathbb{C}}$. Then the following hold.*

(1) If $f : D \to D'$ is a K_1-quasiconformal mapping and if $g : D' \to D''$ is a K_2-quasiconformal mapping, then the composite mapping $g \circ f : D \to D''$ is $K_1 K_2$-quasiconformal.

(2) If $f : D \to D'$ is a K-quasiconformal mapping, then the inverse mapping $f^{-1} : D' \to D$ is also K-quasiconformal.

(3) If $f : D \to D'$ is a 1-quasiconformal mapping, then f is conformal; in other words, f is biholomorphic.

(4) If $f : D \to D'$ is non-constant quasiregular, then $f_z \neq 0$ almost everywhere in D.

The last property together with property (3) implies that for a quasiconformal mapping f, $f(E)$ is of area zero if and only if so is E for a Borel subset E of D. It also enables us to define the Borel measurable function

$$\mu_f = \frac{f_{\bar{z}}}{f_z}$$

for a quasiregular mapping $f : D \to \widehat{\mathbb{C}}$. The measurable function μ_f is called the *Beltrami coefficient* of f. The quantity

$$K(f) = \frac{1 + \|\mu_f\|_\infty}{1 - \|\mu_f\|_\infty}$$

is called the *maximal dilatation* of f, where $\|\mu_f\|_\infty$ denotes the essential sup-norm of μ_f. Note that f is K-quasiregular if and only if $K(f) \leq K$. In particular, $\mu_f = 0$ on a subdomain D_0 of D precisely when f is holomorphic on D_0. The Beltrami coefficients obey the following transformation rule. For a quasiconformal mapping f and a quasiregular mapping g, the formula

$$(\mu_{g \circ f^{-1}} \circ f) \frac{\overline{f_z}}{f_z} = \frac{\mu_g - \mu_f}{1 - \overline{\mu_f} \cdot \mu_g}$$

holds almost everywhere as long as $g \circ f$ is defined.

In particular, we obtain the following result, which is noteworthy in applications of quasiconformal mappings.

Lemma 1.7. *Let $f : D \to D'$ be a quasiconformal mapping and $g : D \to \widehat{\mathbb{C}}$ be a quasiregular mapping with $\mu_f = \mu_g$ almost everywhere on D. Then $h = g \circ f^{-1} : D' \to \widehat{\mathbb{C}}$ is holomorphic.*

The most fundamental result in the theory of plane quasiconformal mapping is the following theorem due to Ahlfors and Bers [AB].

Theorem 1.8 (Measurable Riemann Mapping Theorem). *Let μ be an essentially bounded measurable function on $\mathbb{C}$ with $\|\mu\|_\infty < 1$. Then there exists a unique quasiconformal mapping $f = f^\mu : \mathbb{C} \to \mathbb{C}$ such that $f(0) = 0, f(1) = 1$ and $\mu_f = \mu$ almost everywhere on $\mathbb{C}$. Moreover, if $\mu = \mu_t$ depends holomorphically on a complex parameter t, then $f^{\mu_t}(z)$ is holomorphic in t.*

Interested readers are encouraged to consult standard textbooks [Ahl, LV] on quasiconformal mappings.

Corollary 1.9 (Stoïlow factorization). *Let g be a quasiregular map on a domain $D \subset \widehat{\mathbb{C}}$. Then there is a quasiconformal mapping $f : D \to D'$ and a holomorphic map $h : D' \to \widehat{\mathbb{C}}$ such that $g = h \circ f$.*

Proof. Indeed, define μ to be the Beltrami coefficient μ_g on D and $\mu = 0$ off the domain D. Then $h = g \circ (f^\mu)^{-1}$ is holomorphic on $D' = f^\mu(D)$ by Lemma 1.7. Thus $f = f^\mu|_D$ and h work.

When dealing with quasiconformal mappings, the following property is very useful. For the proof, we refer to [LV, II §5].

Theorem 1.10. *(1) The class of K-quasiconformal mappings $f : \mathbb{C} \to \mathbb{C}$ with $f(0) = 0, f(1) = 1$ is normal in the sense that each sequence in this class contains a subsequence converging locally uniformly on $\mathbb{C}$.*
(2) Let f_n $(n = 1, 2, 3, \dots)$ be a sequence of K-quasiconformal mappings on a domain D which converges to a mapping f locally uniformly on D. Then f is either a K-quasiconformal mapping on D or a constant mapping.

One may think that the notion of quasiconformal mappings is somewhat artificial. However, it arises quite naturally in the context of complex analysis. As evidence, we introduce to the notion of holomorphic motions.

Let X be a domain in $\mathbb{C}^n$ (or more generally, a complex manifold) with basepoint λ_0 and let E be a subset of $\widehat{\mathbb{C}}$. A mapping $H : X \times E \to \widehat{\mathbb{C}}$ is called a *holomorphic motion* of E over X if the following three properties are satisfied:

(i) $H(\lambda_0, z) = z$ for all $z \in E$.
(ii) For each $z \in E$, the mapping $\lambda \mapsto H(\lambda, z)$ is holomorphic on X.
(iii) For each $\lambda \in X$, the mapping $z \mapsto H(\lambda, z) = H_\lambda(z)$ is injective on E.

Note that we do not assume joint continuity of the map $H(\lambda, z)$. Nevertheless, the following strong conclusion can be deduced. Here and in what follows, the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ is denoted by $\mathbb{D}$.

Theorem 1.11 (Mañé-Sad-Sullivan [MSS]). *Let $H : \mathbb{D} \times E$ be a holomorphic motion of E over the unit disk $\mathbb{D}$. Then H extends uniquely to a holomorphic motion of the closure $\bar{E}$ of E over $\mathbb{D}$ in such a way that $H : \mathbb{D} \times \bar{E} \to \widehat{\mathbb{C}}$ is continuous. Moreover, for each $\lambda \in \mathbb{D}$, the mapping $H_\lambda(z) = H(\lambda, z)$ is $|\lambda|$-quasiconformal on each connected component of the interior of $\bar{E}$.*

The above theorem is often called the *λ-lemma*. As a corollary, we have the following characterization of quasiconformal mappings on the complex plane.

Theorem 1.12. *Let $0 < k < 1$. A mapping $f : \mathbb{C} \to \mathbb{C}$ fixing 0 and 1 is k-quasiconformal if and only if there is a holomorphic motion H of $\mathbb{C}$ over $\mathbb{D}$ such that $f(z) = H(k, z)$ for $z \in \mathbb{C}$.*

The "if" part follows from the Mañé-Sad-Sullivan theorem. On the other hand, if f is k-quasiconformal, then we consider the map $H : \mathbb{D} \times \mathbb{C} \to \mathbb{C}$ defined by $H(\lambda, z) = f^{\lambda\mu}(z)$, where $\mu = \mu_f / k$ and $f^{\lambda\mu}$ is constructed by the Measurable Riemann Mapping Theorem. Then H is a holomorphic motion of $\mathbb{C}$ over $\mathbb{D}$ by the Ahlfors-Bers theorem and $f = H_k$ by construction.

Finally, we mention the miraculas result proved by Slodkowski [Slo].

Theorem 1.13. *Any holomorphic motion H of a subset E of $\widehat{\mathbb{C}}$ over $\mathbb{D}$ extends to a holomorphic motion $\tilde{H}$ of $\widehat{\mathbb{C}}$ over $\mathbb{D}$; that is, $\tilde{H}|_{E \times \mathbb{D}} = H$.*

Note that the extension $\tilde{H}$ is not necessarily unique. More refined results can be found in [EKK].

2. Various classes of domains

In Geometric Function Theory, it is important to study specific classes of simply connected domains and their Riemann mapping functions. Here, we recall the Riemann Mapping Theorem.

Theorem 2.1 (Riemann Mapping Theorem). *Let D be a simply connected proper subdomain of $\mathbb{C}$. For any given point $w_0 \in D$, there exists a unique conformal homeomorphism g of D onto the unit disk $\mathbb{D}$ such that $g(w_0) = 0$ and that $g'(w_0)$ is a positive real number.*

In our context, it is important to look at the inverse map $f = g^{-1} : \mathbb{D} \to D$ with $f(0) = w_0$ and $f'(0) > 0$. We call f the *Riemann mapping function* of D (with basepoint w_0). Note that, in the literature, g is often called the Riemann mapping function. By taking the translation $w \to w - w_0$, we may assume $w_0 = 0$ and, indeed, we will do it in what follows. By taking a dilation $w \to w/f'(0)$, we may further normalize so that $f(0) = 0$ and $f'(0) = 1$ but for a while we will not do that.

Typical and important domains are summarized here. Let D be a domain in $\mathbb{C}$ with $0 \in D$. Below, λ and α are given real numbers with $-\pi/2 < \lambda < \pi/2$ and $0 \le \alpha < 1$.

(I) D is called *starlike* (with respect to 0) if the line segment $[0, z_1]$ is contained in D whenever $z_1 \in D$.
(II) D is called *convex* if the line segment $[z_0, z_1]$ is contained in D whenever $z_0, z_1 \in D$.
(III) D is called *close-to-convex* or linearly accessible if the complement $\mathbb{C} \backslash D$ is a union of half-lines which do not intersect except for their tips.
(IV) D is called *λ-spirallike* (with respect to 0) if the λ-spiral segment $[0, z_1]_\lambda$ is contained in D whenever $z_1 \in D$. Here, $[0, z_1]_\lambda = \{z_1 \exp(te^{i\lambda}) : t \in [-\infty, 0]\}$.
(V) D is called *strongly starlike of order* α (with respect to 0) if $z_1 \cdot U_\alpha \subset D$ for all $z_1 \in D$. Here $U_\alpha = \{\exp(te^{i\nu}) : \nu \in (-\pi(1-\alpha)/2, \pi(1-\alpha)/2), t \in [-\infty, 0]\}$.
(VI) D is called *strongly λ-spirallike of order* α (with respect to 0) if $z_1 \cdot U_{\lambda,\alpha} \subset D$ for all $z_1 \in D$. Here $U_{\lambda,\alpha} = \{\exp(te^{i\nu}) : \nu \in (\lambda - \pi(1-\alpha)/2, \lambda + \pi(1-\alpha)/2), t \in [-\infty, 0]\}$.

Obviously, convex domains are starlike and starlike domains are close-to-convex. We also remark that 0-spirallike domains are nothing but starlike domains. Likewise, strongly λ-spirallikeness reduces to strong starlikeness when $\lambda = 0$.

In GFT, it is an important issue to characterize geometric properties of domains in terms of their Riemann mapping functions. We have the following results for the above classes of domains.

Theorem 2.2. *Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function with $f(0) = 0$ and $f'(0) > 0$. Set $D = f(\mathbb{D})$ and suppose numbers $\lambda \in (-\pi/2, \pi/2)$ and $\alpha \in [0, 1)$ are given.*

(I) f is univalent and D is starlike if and only if $\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0$ *on* $\mathbb{D}$.
(II) f is univalent and D is convex if and only if $\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0$ *on* $\mathbb{D}$.
(III) f is univalent and D is close-to-convex if and only if $\operatorname{Re}\left(e^{-i\nu}\frac{zf'(z)}{g(z)}\right) > 0$ *on $\mathbb{D}$ for a $\nu \in (-\pi/2, \pi, 2)$ and a function g satisfying the condition (I) above.*
(IV) f is univalent and D is λ-spirallke if and only if $\operatorname{Re}\left(e^{-i\lambda}\frac{zf'(z)}{f(z)}\right) > 0$ *on* $\mathbb{D}$.
(V) f is univalent and D is strongly starlike of order α if and only if $\left|\arg \frac{zf'(z)}{f(z)}\right| \le \frac{\pi\alpha}{2}$ *on* $\mathbb{D}$.
(VI) f is univalent and D is strongly λ-spirallike of order α if and only if $\left|\arg \frac{zf'(z)}{f(z)} - \lambda\right| \le \frac{\pi\alpha}{2}$ *on* $\mathbb{D}$.

We call the function appearing in one of the above list of conditions by the name of the domain. For instance, we call f in condition (I) a starlike function.

It is remarkable that the above analytic conditions even imply univalence of the functions $f(z)$. For (I)–(IV), see [Dur]. See [Sug05] and [Sug12] for (V) and (VI) respectively.

3. Quasidisks and quasiconformal extension

In the context of quasiconformal mappings, the most important class of simply connected domains is that of quasidisks. Here, a domain D in $\widehat{\mathbb{C}}$ is called a *quasidisk* if $D = f(\mathbb{D})$ for a quasiconformal mapping $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. More specifically, D is called a K-quasidisk if we can take a K-quasiconformal f. Note that a quasidisk is a Jordan domain. But the converse is not true. The boundary of a K-quasidisk is called a *K-quasicircle*.

Here is a strong connection between quasidisks and quasiconformal extension of Riemann mapping functions.

Theorem 3.1. *Let D be a simply connected domain in $\widehat{\mathbb{C}}$ with non-degenerate boundary. A conformal homeomorphism $f : \mathbb{D} \to D$ extends to a quasiconformal mapping of $\widehat{\mathbb{C}}$ if and only if D is a quasidisk.*

Also, the notion of quasiconformal reflection is important. Here, an orientation-reversing homeomorphism $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is called a *K-quasiconfromal reflection* in a Jordan curve J if $z \mapsto h(\bar{z})$ is a K-quasiconformal mapping and if h is an involution (namely, $h \circ h$ =id) fixing J pointwise. The following is also due to Ahlfors (see [Ahl]).

Theorem 3.2. *A Jordan curve J in $\widehat{\mathbb{C}}$ is a quasicircle if and only if there is a quasiconformal reflection in J.*

Kühnau refined this to the assertion (see [GeHa, p. 21]): *J is a K-quasicircle if and only if it admits a K-quasiconformal reflection in J.*

It is, in general, a hard problem to find the least K for a given domain to be a K-quasidisk. For instance, it is known (see [GeHa, p. 12]) that the sector $S(\alpha) = \{z : |\arg z| < \alpha/2\}$

is a K_α-quasidisk and K_α is sharp for $0 < \alpha < 2\pi$, where

$$K_\alpha = \max\left\{\sqrt{\frac{2\pi - \alpha}{\alpha}}, \sqrt{\frac{\alpha}{2\pi - \alpha}}\right\}.$$

There are metric characterizations of quasidisks. The following one is found in [Ahl].

Theorem 3.3 (Ahlfors three-point property). *A Jordan curve C in $\widehat{\mathbb{C}}$ passing through ∞ is a quasicircle if and only if there is a constant $A \geq 1$ such that the inequality*

$$\frac{|z_1 - z^*|}{|z_1 - z_2|} \leq A$$

holds for any points z_1, z^*, z_2 on C situated in this order.

The above condition means that the diameter of the subarc of C joining z_1 and z_2 in $\mathbb{C}$ is bounded by a constant multiple of $|z_1 - z_2|$. For a Jordan curve which does not pass through ∞, the so-called "bounded turning" condition is useful. See [GeHa, p. 23] for instance.

Theorem 3.4. *A Jordan curve C in $\mathbb{C}$ is a quasicircle if and only if there is a constant $A \geq 1$ such that the inequality $\min\{\operatorname{diam} C_1, \operatorname{diam} C_2\} \leq A|z_1 - z_2|$ holds for each pair of distinct points z_1, z_2 in C, where C_1 and C_2 are the connected components of $C\backslash\{z_1, z_2\}$.*

Note that bounded starlike domains are not necessarily quasidisks (indeed, some of them are not Jordan domains).

The following characterization is due to Gehring [Geh]. We use here the symbols $\mathbb{D}(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$ and $\overline{\mathbb{D}}(a, r) = \{z \in \mathbb{C} : |z - a| \leq r\}$.

Theorem 3.5. *Let D be a simply connected domain in $\mathbb{C}$. Then D is a quasidisk if and only if the following two conditions are satisfied:*

(i) There exists a constant $a > 1$ such that for all $z_0 \in \mathbb{C}$ and $r > 0$ any two points in $D \cap \overline{\mathbb{D}}(z_0, r)$ can be joined by an arc in $D \cap \overline{\mathbb{D}}(z_0, ar)$.

(ii) There exists a constant $0 < b < 1$ such that for all $z_0 \in \mathbb{C}$ and $r > 0$ any two points in $D\backslash\mathbb{D}(z_0, r)$ can be joined by an arc in $D\backslash\mathbb{D}(z_0, br)$.

After the work of Gehring, this sort of notions has been intensively studied and extended to more general domains. For instance, the above two conditions are studied separately. Domains with Property (i) are called *linearly connected* and (bounded) domains with Property (ii) are called *John domains*. It is known that bounded linearly connected domains are Jordan domains. See [Pom92] for details.

There are many other characterizations of quasidisks. See the monograph [GeHa] by Gehring and Hag for more information.

It is an important and interesting problem to give a sufficient condition for a simply connected domain to be a quasidisk. Equivalently, we want to have a sufficient condition for a univalent function on the unit disk to have a quasiconformal extension to the whole sphere $\widehat{\mathbb{C}}$. Here are a couple of results in this line.

Theorem 3.6. *A strongly λ-spirallike function of order α extends to a $\sin(\pi\alpha/2)$-quasiconformal mapping of $\mathbb{C}$.*

The case when $\lambda = 0$ (namely, the strongly starlike case) was proved by Fait, Krzyż and Zygmunt [FKZ]. The general case was shown by Sevodin [Sev] (see also [Sug12]).

We outline the proof in the case when $\lambda = 0$. Let D be a domain containing 0 and define the function

$$R_D(\theta) = \sup\{r > 0 : [0, re^{i\theta}] \subset D\}$$

for $\theta \in \mathbb{R}$. The following result gives another characterization of strongly starlike domains (see [Sug05]).

Theorem 3.7. *A domain D with $0 \in D$ is strongly starlike of order α if and only if R_D is absolutely continuous on $[0, 2\pi]$ and it satisfies the inequality*

$$\frac{|R_D'(\theta)|}{R_D(\theta)} \leq \tan\frac{\pi\alpha}{2}$$

for almost every θ.

In particular, a strongly starlike domain D is a bounded Jordan domain because the mapping $e^{i\theta} \mapsto R_D(\theta)e^{i\theta}$ gives an injective parametrization of the boundary of D.

For instance, we see that the rectangle $|x| < a, |y| < b$ with $a \geq b$ is strongly starlike of order $(2/\pi)\arctan(a/b)$. In particular, the square $\max\{|x|, |y|\} < a$ is strongly starlike of order $1/2$.

By using the above theorem, we construct the quasiconformal extension of a strongly starlike function f of order α as follows. Let $D = f(\mathbb{D})$. We first note that the function f extends to a homeomorphism $f : \overline{\mathbb{D}} \to \overline{D}$. Then we define a quasiconformal reflection in ∂D by

$$h(re^{i\theta}) = \frac{R_D(\theta)^2}{r}e^{i\theta}.$$

Note that h swaps 0 and ∞. By Theorem 3.7, we can see that h is indeed a $\sin(\pi\alpha/2)$-quasiconformal reflection. Then we can now give a $\sin(\pi\alpha/2)$-quasiconformal extension of f by

$$\tilde{f}(z) = \begin{cases} f(z) & \text{if } |z| \leq 1, \\ h(f(1/\bar{z})) & \text{if } |z| > 1. \end{cases}$$

Similar and more general results are found in [KVW].

Theorem 3.8. *Let D be a convex domain such that $\mathbb{D}(a, r_0) \subset D \subset \mathbb{D}(a, r_1)$ for some $a \in \mathbb{C}$ and $0 < r_0 \leq r_1 < +\infty$. Then D is a strongly starlike domain of order $(2/\pi)\arccos(r_0/r_1)$. In particular, D is a $\frac{1+k}{1-k}$-quasidisk, where $k = \sqrt{1-(r_0/r_1)^2}$.*

Proof. We may assume that $a = 0$. Let $f : \mathbb{D} \to D$ be a Riemann map with $f(0) = 0, f'(0) > 0$. Keeping Theorem 1.10 in mind, by the standard approximation $f_r(z) = f(rz)/r$ for $0 < r < 1$, we may assume that the boundary of D is smooth. (Note that $f(\mathbb{D}(0, r))$ is convex for $0 < r < 1$ by a theorem of Study, which also follows from the analytic characterization of convex domains.) First we consider the boundary point $z = x_0 = R_D(0)$ of D on the positive real axis. Suppose that the tangent line at x_0 of ∂D has the form $x = x_0 + my$ for a constant $m \in \mathbb{R}$. Since the tangent line does not intersect the inner circle $|z| = r_0$ and $x_0 \leq r_1$, we have $|m| \leq \tan(\arccos(r_0/r_1)) =: M$. Therefore, $|R_D(\theta) - R_D(0)| \leq |m|R_D(0)|\theta| + o(\theta)$ as $\theta \to 0$. This, in turn, implies that $|R'_D(0)| \leq |m|R_D(0) \leq MR_D(0)$. By rotating, we obtain the inequality $|R'_D(\theta)| \leq MR_D(\theta)$ for any $\theta \in \mathbb{R}$. Theorem 3.7 now implies that D is strongly starlike of order $\alpha = (2/\pi)\arccos(r_0/r_1)$. We apply Theorem 3.6 with $\lambda = 0$ to obtain that D is a $\frac{1+k}{1-k}$-quasidisk, where $k = \sin(\pi\alpha/2) = \sqrt{1-(r_0/r_1)^2}$.

This result is also found in [KVW]. Note that an unbounded convex domain might not be a quasidisk. For instance, the pararell strip $|\operatorname{Im} z| < 1$ is not a quasidisk because it is not a Jordan domain in $\widehat{\mathbb{C}}$.

In connection with univalence and quasiconformal extension criteria, Schwarzian and pre-Shwarzian derivatives are very important. For a non-constant meromorphic function f on a domain, we define

$$T_f = \frac{f''}{f'} \quad \text{and} \quad S_f = T'_f - \frac{1}{2}T_f^2.$$

We now consider the hyperbolic norm of weight j for an analytic function φ on $\mathbb{D}$ by

$$\|\varphi\|_j = \sup_{z \in \mathbb{D}} (1-|z|^2)^j |\varphi(z)|.$$

Then the following result is well known.

Theorem 3.9. *(i) (Becker, Becker-Pommerenke) Let f be a non-constant holomorphic function on $\mathbb{D}$. If f is univalent on $\mathbb{D}$ then $\|T_f\|_1 \leq 6$. On the other hand, if $\|T_f\|_1 \leq 1$, then f is univalent on $\mathbb{D}$. Moreover, if $\|T_f\|_1 \leq k < 1$, then f extends to a k-quasiconformal mapping of $\mathbb{C}$.*

(ii) (Kraus, Nehari, Ahlfors-Weill) Let f be a non-constant meromorphic function on $\mathbb{D}$. If f is univalent on $\mathbb{D}$ then $\|S_f\|_2 \leq 6$. On the other hand, if $\|S_f\|_2 \leq 2$, then f is univalent on $\mathbb{D}$. Moreover, if $\|S_f\|_2 \leq 2k < 2$, then f extends to a k-quasiconformal mapping of $\widehat{\mathbb{C}}$.

For more results in this line, the reader may refer to [Sug07] and [Hot].

4. Coefficient problems related to quasiconformal extension

In Geometric Function Theory, it is important to consider extremal problems. To clarify the range of considerations, we need to set up suitable classes of functions. Concerning the quasiconformal extension problem, we define classes as follows. Let $\mathcal{A}$ denote the set of analytic functions f on $\mathbb{D}$ normalized so that $f(0) = 0$ and $f'(0) = 1$. Hence, a function $f \in \mathcal{A}$ can be expanded in the convergent power series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n = z + a_2 z^2 + a_3 z^3 + \cdots \qquad (|z| < 1). \quad (4.1)$$

We denote by $\mathcal{S}$ the subset of $\mathcal{A}$ consisting of univalent (= injective) functions on $\mathbb{D}$. The famous Bieberbach conjecture (now known as the de Branges theorem) asserts that $|a_n| \leq n$ for $n = 2, 3, 4, \ldots$ for $f \in \mathcal{S}$ and equality holds for some $n \geq 2$ precisely when f is the Koebe function $K(z) = z/(1-z)^2 = z + 2z^2 + 3z^3 + \cdots$ or its rotation $e^{-i\theta}K(e^{i\theta}z)$. We next consider two kinds of subclasses of $\mathcal{S}$ for $0 \leq k < 1$: Denote by $\mathcal{S}(k)$ (resp. $\mathcal{S}^*(k)$) the set of functions $f \in \mathcal{S}$ which extend to k-quasiconformal mappings of $\widehat{\mathbb{C}}$ (resp. $\mathbb{C}$). $f \in \mathcal{S}^*(k)$ means that f extend to a k-quasiconformal mapping of $\widehat{\mathbb{C}}$ which fixes ∞. Therefore, $\mathcal{S}^*(k) \subset \mathcal{S}(k)$. For instance, we have the following result.

Proposition 4.1. *Let $f \in \mathcal{S}$ and set $f_r(z) = f(rz)/r$ for $0 < r < 1$. Then $f_r \in \mathcal{S}^*(r) \cap \mathcal{S}(r^2)$.*

To show that $f \in \mathcal{S}^*(r)$, we consider the map $H : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ defined by $H(\lambda, z) = f(\lambda z)/\lambda$ for $\lambda \neq 0$ and $H(0, z) = z$. Then H is a holomorphic motion of $\mathbb{D}$ over $\mathbb{D}$. Extend H to $\mathbb{D} \cup \{\infty\}$ by setting $H(\lambda, \infty) = \infty$. We now apply the Slodkowski theorem (Theorem 1.13) to obtain a holomorphic motion $\tilde{H}$ of $\widehat{\mathbb{C}}$ over $\mathbb{D}$ such that $\tilde{H}(\lambda, z) = H(\lambda, z)$ for $\lambda \in \mathbb{D}$ and $z \in \mathbb{D} \cup \{\infty\}$. By Theorem 1.11, it implies that $f_r(z) = H(r, z)$ extends to the r-quasiconformal mapping $\tilde{H}(r, z)$ of $\mathbb{C}$. Hence, we conclude that $f_r \in \mathcal{S}^*(r)$. The proof of the assertion $f_r \in \mathcal{S}(r^2)$ is contained in Example 2.1 in [Sug99].

We now consider the extremal problems on the coefficient $a_n = a_n(f)$ of f in (4.1). We suggest the reader to consult the survey paper [Hot] for more details. Let

$$A(n, k) = \sup_{f \in \mathcal{S}(k)} |a_n(f)| \quad \text{and} \quad A^*(n, k) = \sup_{f \in \mathcal{S}^*(k)} |a_n(f)|$$

for $n \geq 2$ and $0 \leq k < 1$. Since $\mathcal{S}^*(k) \subset \mathcal{S}(k)$, we have $A^*(n, k) \leq A(n, k)$. By the above proposition and the de Branges theorem, we also have

$$\lim_{k \to 1^-} A^*(n, k) = \lim_{k \to 1^-} A(n, k) = n.$$

We observe that $A(n, k)$ and $A^*(n, k)$ are both non-decreasing in $0 \leq k < 1$. Note that the classes $\mathcal{S}(k)$ and $\mathcal{S}^*(k)$ are both compact so that the suprema can be replaced by maxima in the above. On the other hand, the Möbius map $T(z) = z/(1-z) = z + z^2 + z^3 + \cdots$ has natural conformal extension to $\widehat{\mathbb{C}}$ and therefore $T \in S(0)$. Hence $A(n, k) \geq 1$ for $0 \leq k < 1$ and $n \geq 2$. Determination of these quantities is largely open except for $n = 2$.

Theorem 4.2.

$$A(2, k) = 2 - 4\left(\frac{\arccos k}{\pi}\right)^2 \quad \text{and} \quad A^*(2, k) = 2k.$$

The first equality is due to Schiffer and Schober [SS]. The second equality is due to Kühnau [Küh69] and the extremal function is given by

$$\Phi(z) = \frac{z}{(1 - kz)^2} = K_k(z) = \sum_{n=1}^{\infty} k^{n-1} n z^n,$$

where $K(z) = z/(1 - z)^2$ is the Koebe function. Indeed, this function has a k-quasiconformal extension to $\mathbb{C}$ of the form

$$\tilde{\Phi}(z) = \frac{z}{(1 - kz/|z|)^2} \quad (|z| > 1).$$

One may expect extremal functions for $A^*(n, k)$ may be the above Φ or its power transform

$$\Phi_m(z) = F(z^m)^{1/m} = \frac{z}{(1 - kz^m)^{2/m}} = z + \frac{2k}{m} z^{m+1} + O(z^{2m+1})$$

for some integer $m > 1$. Taking $m = n - 1$, we have the first estimate of the following result.

Theorem 4.3. *For $n \geq 2$ and $0 \leq k < 1$,*

$$\frac{2k}{n-1} \leq A^*(n, k) \leq kn. \tag{4.2}$$

We show now the inequality $A^*(n, k) \leq kn$. Let $f \in \mathcal{S}^*(k)$ for some $0 < k < 1$. By definition, there is a k-quasiconformal mapping $\tilde{f}$ of $\mathbb{C}$ onto itself such that $\tilde{f}|_{\mathbb{D}} = f$. Put $\mu = \mu_{\tilde{f}}/k$ and consider the normalized quasiconformal mapping $f^{\lambda\mu}$ in Theorem 1.8 for $\lambda \in \mathbb{D}$. Since it is conformal on $\mathbb{D}$, we can expand it in the form

$$f^{\lambda\mu}(z) = \sum_{n=1}^{\infty} c_n(\lambda) z^n.$$

Note that each $c_n(\lambda)$ is a holomorphic function in $|\lambda| < 1$. Then the function

$$H_\lambda(z) = f^{\lambda\mu}(z)/c_1(\lambda) = z + \sum_{n=2}^{\infty} a_n(\lambda) z^n$$

belongs to $\mathcal{S}$. Now the de Branges theorem yields the estimate $|a_n(\lambda)| \leq n$ for $\lambda \in \mathbb{D}$. Since $H_0 = \mathrm{id}$, we have $a_n(0) = 0$. Hence Schwarz's lemma implies that $|a_n(\lambda)| \leq |\lambda| n$. In particular, we have $|a_n| = |a_n(k)| \leq kn$ as required.

Krushkal [Kru] claimed that equality holds in the first inequality in (4.2) for sufficiently small k. However, the following recent result revealed that the claim is false at least when $n = 3$.

Theorem 4.4 (Gumenyuk and Hotta [GuHo]).

$$\begin{aligned} k(1 + e^{1-1/k}(1 + k)) &< A^*(3, k) \\ &\leq \min_{0 \leq \alpha \leq 1} \left[(1 + 2e^{-2\alpha/(1-\alpha)})k + 4\alpha k^2 \right]. \end{aligned}$$

As a counterpart of the class $\mathcal{S}$, the class Σ has been studied for a long time. Here, Σ denotes the set of univalent analytic functions $F(z)$ on $|z| > 1$ which have the Laurent series expansion of the form

$$F(z) = z + \sum_{n=0}^{\infty} b_n z^{-n} = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots \qquad (|z| > 1).$$

We know sharp inequalities $|b_1| \le 1$, $|b_2| \le 2/3$ and $|b_3| \le e^{-6} + 1/2$. See for instance [Dur]. However, up to now, sharp bounds are not known for $b_n,\ n \ge 4$. Correspondingly to $\mathcal{S}(k)$ and $\mathcal{S}^*(k)$, we consider the following subclasses of Σ. Denote by $\Sigma(k)$ (resp. $\Sigma^*(k)$) the set of functions $F \in \Sigma$ which extend to k-quasiconformal mappings of $\widehat{\mathbb{C}}$ (resp. $\widehat{\mathbb{C}} \backslash \{0\}$).

Note that for $f \in \mathcal{S}(k)$, the function $F(z) = 1/f(1/z)$ belongs to $\Sigma(k)$. However, for $F \in \Sigma(k)$ the function $f(z) = 1/F(1/z)$ does not necessarily belong to $\mathcal{S}(k)$ because f may have a pole in $\mathbb{D}$. If $F(z) \ne 0$ for $|z| > 1$, then this f belongs to $\mathcal{S}(k)$. On the other hand, when $F(z) = 1/f(1/z)$, $f \in \mathcal{S}^*(k)$ if and only if $F \in \Sigma^*(k)$. We further define

$$B(n,k) = \sup_{F \in \Sigma(k)} |b_n(F)| \quad \text{and} \quad B^*(n,k) = \sup_{f \in \Sigma^*(k)} |b_n(F)|$$

for $n \ge 0$ and $0 \le k < 1$. Since $\Sigma^*(k) \subset \Sigma(k)$, we have $B^*(n,k) \le B(n,k)$. By the Lehto theorem [Leh] we have

$$\sum_{n=1}^{\infty} n |b_n|^2 \le k^2$$

for $F \in \Sigma(k)$. Therefore, we have the crude estimate

$$B^*(n,k) \le B(n,k) \le \frac{k}{\sqrt{n}}$$

for $n \ge 1$. Kühnau [Küh69] showed that $B^*(0,k) = 2k$ and that $B^*(1,k) = B(1,k) = k$. Indeed, extremal functions are given by $1/\Phi(1/z)$ and $1/\Phi_2(1/z)$, respectively. On the other hand, the constant term may be arbitrary for functions in $\Sigma(k)$, we have no bound for $B(0,k)$. It is a challenging problem to determine, or estimate, the quantities $B(n,k)$ and $B^*(n,k)$ for even small n.

5. Grunsky coefficients

Let $F(\zeta) = \zeta + b_0 + b_1 \zeta^{-1} + \cdots$ be a holomorphic function on $|\zeta| > R$ for some $R \ge 1$. Then we look at the form

$$\frac{F(\zeta) - F(\omega)}{\zeta - \omega} = 1 + \sum_{n=1}^{\infty} b_n \frac{\zeta^{-n} - \omega^{-n}}{\zeta - \omega}.$$

Since

$$\frac{\zeta^{-1} - \omega^{-1}}{\zeta - \omega} = -\frac{1}{\zeta\omega},$$

we compute

$$\frac{F(\zeta) - F(\omega)}{\zeta - \omega} = 1 - \sum_{n=1}^{\infty} b_n \left(\zeta^{-n}\omega^{-1} + \zeta^{1-n}\omega^{-2} + \cdots + \zeta^{-1}\omega^{-n}\right).$$

Thus we can expand in the form

$$\log \frac{F(\zeta) - F(\omega)}{\zeta - \omega} = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\beta_{m,n}}{\zeta^m \omega^n}$$

in $|\zeta| > R, |\omega| > R$. The coefficients $\beta_{m,n}$ are called the *Grunsky coefficients* of F. Note that $\beta_{m,n} = \beta_{n,m}$. We write

$$F(\zeta) = \zeta + \sum_{n=0}^{\infty} b_n \zeta^{-n} = \zeta + b_0 + G(\zeta).$$

Then $G(\zeta) \to 0$ as $\zeta \to \infty$. Therefore,

$$\log \frac{F(\zeta) - F(\omega)}{\zeta - \omega} = \log\left(1 + \frac{G(\zeta) - G(\omega)}{\zeta - \omega}\right) = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\beta_{m,n}}{\zeta^m \omega^n}.$$

Noting that for a fixed ζ

$$\log\left(1 + \frac{G(\zeta) - G(\omega)}{\zeta - \omega}\right) = \frac{G(\zeta) - G(\omega)}{\zeta - \omega} + o(\omega^{-1})$$

as $\omega \to \infty$, we have

$$-G(\zeta) = \lim_{\omega \to \infty} \omega \log\left(1 + \frac{G(\zeta) - G(\omega)}{\zeta - \omega}\right) = -\sum_{m=1}^{\infty} \frac{\beta_{m,1}}{\zeta^m}.$$

Hence we now have

$$b_m = \beta_{m,1}, \quad m \ge 1.$$

The following inequality had been a powerful tool to attack the Bieberbach conjecture for a long time. For the proof, we refer to [Pom74] or [Dur].

Theorem 5.5 (The strong Grunsky inequality). *A function $F(\zeta)$ as before is analytic and univalent on $|\zeta| > 1$ if and only if the inequality*

$$\sum_{m=1}^{\infty} m \left| \sum_{n=1}^{\infty} \beta_{m,n} x_n \right|^2 \le \sum_{n=1}^{\infty} \frac{|x_n|^2}{n}$$

holds for every complex sequence $\{x_n\}$ as long as the right-hand side is convergent.

Since $b_m = \beta_{m,1}$ for $F(\zeta) = \zeta + \sum_{m=0}^{\infty} b_m \zeta^{-m}$ in Σ, by choosing $x_n = \delta_{1,n}$, we obtain the inequality

$$\sum_{m=1}^{\infty} m |b_m|^2 \le 1,$$

which is known as Gronwall's area theorem. The strong Grunsky inequality is known to be equivalent to the (original) Grunsky inequality (see [Pom74])

$$\left|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\beta_{m,n}x_m x_n\right| \le \sum_{n=1}^{\infty}\frac{|x_n|^2}{n}.$$

By making a change of variables $x_m/\sqrt{m} = z_m$, we have the inequality

$$\left|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sqrt{mn}\beta_{m,n}z_m z_n\right| \le \sum_{n=1}^{\infty}|z_n|^2.$$

In particular,

$$|\beta_{m,n}| \le \frac{1}{\sqrt{mn}}, \quad m,n \ge 1.$$

The operator $\mathcal{G}[F] : \ell^2 \to \ell^2$ defined by the symmetric matrix $(\sqrt{mn}\beta_{m,n})_{m,n}$ is called the *Grunsky operator*. The above inequality means that $\|\mathcal{G}[F]\|_{\ell^2} \le 1$. The operator norm $\kappa(F) = \|\mathcal{G}[F]\|_{\ell^2}$ is called the *Grunsky constant*. Thus F is univalent on $|\zeta| > 1$ if and only if $\kappa(F) \le 1$. For relationship with quasiconformal mappings, we can state the following. See, for example, [Küh82] for details.

Theorem 5.6. *If $F \in \Sigma(k)$, then $\kappa(F) \le k$. Conversely, if $\kappa(F) < 1$, then $F \in \Sigma(k')$ for some $k' < 1$.*

In particular, for $F \in \Sigma(k)$,

$$|\beta_{m,n}| \le \frac{k}{\sqrt{mn}}, \quad m,n \ge 1.$$

It is a challenging problem to find the sharp bound for $\beta_{m,n}$ among the class $\Sigma(k)$ (even for the case $k = 1$).

Now we turn to the class $\mathcal{S}$ and consider the similar expansion

$$\log\frac{f(z)-f(w)}{z-w} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\gamma_{m,n}z^m w^n, \quad z,w \in \mathbb{D},$$

for $f \in \mathcal{S}$. The numbers $\gamma_{m,n}$ are called the Grunsky coefficients of f. We note that $\gamma_{0,0} = 0$ and, setting $w = 0$, we get

$$\log\frac{f(z)}{z} = \sum_{m=1}^{\infty}\gamma_{m,0}z^m = \sum_{m=1}^{\infty}\gamma_m z^m.$$

Here γ_m are called the logarithmic coefficients of f which played an important role in the proof of the Bieberbach conjecture. We now set $F(\zeta) = 1/f(1/\zeta)$ and compute

$$\log\frac{F(\zeta)-F(\omega)}{\zeta-\omega} = \log\frac{f(z)-f(w)}{z-w} - \log\frac{f(z)}{z} - \log\frac{f(w)}{w}$$

with $z = 1/\zeta$ and $w = 1/\omega$. Hence we have the relations $\beta_{m,n} = -\gamma_{m,n}$ for $m,n \ge 1$ for the Grunsky coefficients.

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H^p-theory for quasiregular mappings

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Abstract. This is a short overview of an important part of H^p-theory for quasiconformal mappings in space. The theorem of Hardy and Littlewood that characterizes H^p-functions in terms of the nontangential maximal function was extended to quasiconformal mappings in space. We consider its possible extensions to quasiregular maps of bounded multiplicity and show that for the case of proper quasiregular maps the theorem holds. A particular emphasis will be given to recent results in this vibrant research area.

Keywords. H^p-spaces, Harmonic mappings, Quasiregular mappings.

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1. Introduction

Quasiregular mappings in $\mathbb{R}^n$ are generalization of the notion of an analytic functions in the complex plane. The class of injective quasiregular mappings is the same as the class of sense-preserving quasiconformal mappings. The basic H^p-theory of quasiconformal mappings in the space was laid down in papers of Zinsmeister [12] and Astala and Koskela [3]. While the H^p-theory of analytic functions is quite rich, the powerful machinery of the plane is not available in the space. Thus, the approach taken in [12] and [3] is to rely on a combination of analytic and geometric aspects of the theory of quasiconformal mappings and a number of tools from the harmonic analysis. Our goal here is to give a brief overview of this theory and point out towards its possible extensions to quasiregular maps with bounded multiplicity or some other additional property as well as to present some of the most recent results in this area.

2. Preliminaries

We need to recall some notions and estimates which all can be found in [4], [9] and [10]. We write $B(x,r)$ for the open ball in $\mathbb{R}^n$ of radius r and centered at x, and we abbreviate $B(0,r)$ to $B(r)$ and $B(0,1)$ to B^n. We denote the boundary of $B(x,r)$ by $S^{n-1}(x,r)$, we write $S^{n-1} = S^{n-1}(0,1)$, and we denote the surface area of S^{n-1} by ω_{n-1}.

The modulus of a family Γ of paths in $\mathbb{R}^n$ is by definition

$$M(\Gamma) = \inf \int_{\mathbb{R}^n} \rho^n dx$$

where the infimum is taken over non-negative Borel functions ρ on $\mathbb{R}^n$ with $\int_\gamma \rho\, ds \geq 1$ for each locally rectifiable $\gamma \in \Gamma$.

A path family is the family Γ of radial segments joining $S^{n-1}(0,r)$, $0 < r < 1$, to a set $E \subset S^{n-1}$. We have that

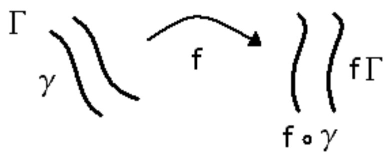
$$M(\Gamma) = \sigma(E)(\log(1/r))^{1-n},$$

where $\sigma(E)$ is the surface area of E. As for upper bounds, we always have

$$M(\Gamma) \leq \frac{\omega_{n-1}}{[\log(R/r)]^{n-1}},$$

if each $\gamma \in \Gamma$ joins $S^{n-1}(x,r)$ to $S^{n-1}(x,R)$, $0 < r < R$.

A homeomorphism of a domain Ω in $\mathbb{R}^n$ into $\mathbb{R}^n$ is K-quasiconformal if f belongs to the Sobolev class $W^{1,n}_{\text{loc}}(\Omega;\mathbb{R}^n)$ and $|Df(x)|^n \leq KJ_f(x)$ for almost every $x \in \Omega$. It then follows [10] that $M(\Gamma)/K \leq M(f\Gamma) \leq K^{n-1}M(\Gamma)$ for all path families $\Gamma \subset \Omega$; here $f\Gamma = \{f \circ \gamma : \gamma \in \Gamma\}$.



Consider a homeomorphism $f : \Omega \to \Omega'$. Suppose that $x \in \Omega$, $x \neq \infty$ and $f(x) \neq \infty$. For each $r > 0$ such that $S^{n-1}(x,r) \subset \Omega$ we set

$$L(x,f,r) = \max_{|y-x|=r} |f(y) - f(x)|,$$

$$l(x,f,r) = \min_{|y-x|=r} |f(y) - f(x)|. \qquad (1)$$

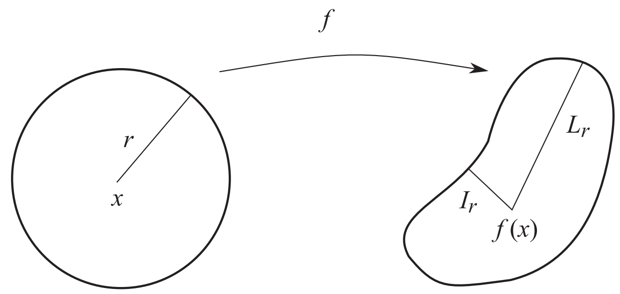
Definition 1. *The linear dilatation of f at x is the number*

$$H(x,f) = \limsup_{r\to 0} \frac{L(x,f,r)}{l(x,f,r)}.$$

If $x = \infty$, $f(x) \neq \infty$, we define $H(x,f) = H(0, f \circ u)$ where u is the inversion $u(x) = \frac{x}{|x|^2}$. If $f(x) = \infty$, we define $H(x,f) = H(x, u \circ f)$.

Example 1. *The mapping $f : \mathbb{B}^n \to \mathbb{B}^n$, $f(x) = |x|^{\alpha-1}x$ has $H(0,f) = 1$.*

This is true for all radial mappings.



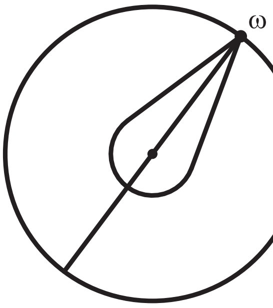
If x denotes a generic point in B^n, and ω in S^{n-1}, we define

$$f(\omega) = \lim_{r\to 1} f(r\omega)$$

whenever this limit exists.

For each $\omega \in S^{n-1}$ we let

$$\Gamma(\omega) = \{x \in B^n : |x - \omega| \leq 3(1 - |x|)\}$$



be the cone with vertex ω. It is easy to see that $S(x) = \{\omega \in S^{n-1} : x \in \Gamma(\omega)\}$: With

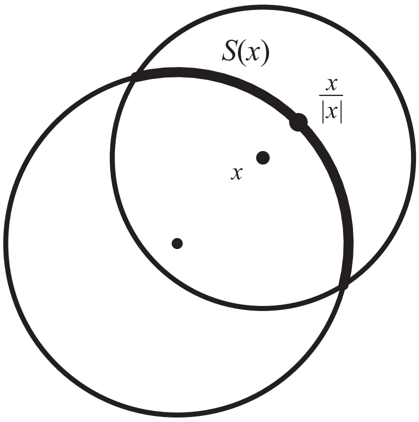
$$T(x) = \{\omega \in S^{n-1} \mid x \in \Gamma(\omega)\}$$

$$\Gamma(\omega) = \{x \in B^n \mid |x - \omega| \leqslant 3(1 - |x|)\}$$

$$S(x) = S^{n-1} \cap B(x, 3(1 - |x|)),$$

we have the following,

$$\begin{aligned} \omega \in S(x) &\Leftrightarrow (\omega \in S^{n-1} \wedge |x - \omega| \leqslant 3(1 - |x|)) \\ &\Leftrightarrow (\omega \in S^{n-1} \wedge \omega \in B^n \wedge |x - \omega| \leqslant 3(1 - |x|)) \\ &\Leftrightarrow (\omega \in S^{n-1} \wedge \omega \in \Gamma(\omega)) \\ &\Leftrightarrow \omega \in \Gamma(\omega). \end{aligned}$$



3. Quasiconformal mappings and H^p-classes

We say that a quasiconformal mapping f of B^n, $n \geq 2$, belongs to the class H^p if

$$\|f\|_{H^p} = \sup_{0<r<1} \left(\int_{S^{n-1}} |f(r\omega)|^p \, d\sigma \right)^{1/p} < \infty.$$

According to the following theorem of Jerison and Weitsman [6], each quasiconformal mapping f belongs to some H^p-class.

Theorem 1 (Jerison-Weitsman). *There exists a constant $p_0 = p_0(n, K) > 0$ so that every K-quasiconformal mapping f of B^n belongs to H^p whenever $p < p_0$.*

By the classical theorem of Prawitz [8], all conformal mappings f of the unit disk belong to H^p for $p < 1/2$, and the Koebe mapping $f(z) = z/(1 - z)^2$ shows that this bound is sharp. The exponent p_0 obtained by Jerison and Weitsman is not the best possible. Astala and Koskela give a new proof for Theorem 1 that yields the sharp exponent in the plane. In higher dimensions, their estimate is optimal for mappings into a half space, but it is still open if the given bound is also best possible in the general situation.

For each $K \geq 1$ and $n \geq 2$, let $a(n, K)$ be the infimum of the numbers a such that

$$\sup_{|x|<1} (1 - |x|)^a |f(x)| < \infty$$

for every K-quasiconformal mapping f of B^n. Then we have:

Theorem 2. *The best possible bound $p_0(n, K)$ in Theorem 1 is*

$$p_0(n, K) = (n - 1)/a(n, K).$$

In particular, $p_0(2, K) = 1/(2K)$, and for $n \geq 3$

$$(n - 1)/(2K)^{1/(n-1)} \leq p_0(n, K) \leq (n - 1)/K^{1/(n-1)}.$$

Moreover, for the subclass of f mapping into a half space,

$$p_0(n, K) = (n - 1)/K^{1/(n-1)}.$$

Recall that μ is a Carleson measure if

$$\mu(B^n \cap B(\omega, \rho)) \leq C\rho^{n-1}$$

for $\omega \in S^{n-1}$ and $\rho > 0$.

4. Zinsmeister's theorem and its extensions

One of the cornerstones of the modern development of H^p-spaces is the theorem of Hardy and Littlewood that characterizes H^p-functions in terms of the nontangential maximal function

$$f^*(\omega) = \sup_{x \in \Gamma(\omega)} |f(x)|, \quad \omega \in S^{n-1}.$$

According to this result a (holomorphic) function f of the disk belongs to H^p if and only if $f^* \in L^p(S^1)$. In [12] Zinsmeister has extended this maximal function characterization to quasiconformal mappings in the space as follows.

Theorem 3 (Zinsmeister). *The following conditions are equivalent for each quasiconformal mapping f of B^n, $n \geq 2$, and for all $0 < p < \infty$.*

1. $f(\omega) \in L^p(S^{n-1})$.
2. $f(x) \in H^p$.
3. $f^*(\omega) \in L^p(S^{n-1})$.

In addition, the corresponding "norms" are equivalent with constants depending only on n, K, p.

Here $f(\omega)$ denotes the radial limit of f at $\omega \in S^{n-1}$ whenever it exists. The original proof of Theorem 3 in [12] was based on a result of Jones [7] on Carleson measures and quasiconformal mappings. Astala and Koskela [3] provided a different approach which is directly tied to the geometric nature of quasiconformal mappings. Their approach is given in the following Lemma and its Corollary whose proofs we sketch here for reader's convenience.

Lemma 1 (Astala-Koskela). *Suppose f is quasiconformal with $f(x) \neq 0$ for all $x \in B^n$. Then, for each $x \in B^n$ and all $M > 1$,*

$$\sigma(\{\omega \in S(x) : |f(\omega)| < |f(x)|/M\}) \leq C(n,K)\sigma(S(x))(\log M)^{1-n}. \qquad (2)$$

Proof. Let us first consider the case where in (2) we have $x = 0$. We may assume that $d(f(0), f(S^{n-1})) = 1$. After this normalization it follows from a simple modulus estimate that $|f(x) - f(0)| \leq 1/2$ for $|x| \leq r_0$; here r_0 depends only on n, K. As

$$1 = d(f(0), f(S^{n-1})) \leq |f(0)|,$$

the set $fB(0, r_0)$ cannot intersect $B(0, |f(0)|/2)$.

Let next $E = \{\omega \in S^{n-1} : |f(\omega)| < |f(0)|/M\}$ and choose Γ_E to be the path family of radial segments connecting $B(0, r_0)$ to E. Then Γ_E has modulus

$$M(\Gamma_E) = \sigma(E)\log(1/r_0)^{1-n}.$$

If $M > 3$, the paths in the image family $\Gamma'_E = f\Gamma_E$ connect the complement of $B(0, |f(0)|/2)$ to $B(0, |f(0)|/M)$ and therefore

$$M(\Gamma'_E) \leq \omega_{n-1}(\log(M/2))^{1-n} \leq C(n)(\log M)^{1-n}.$$

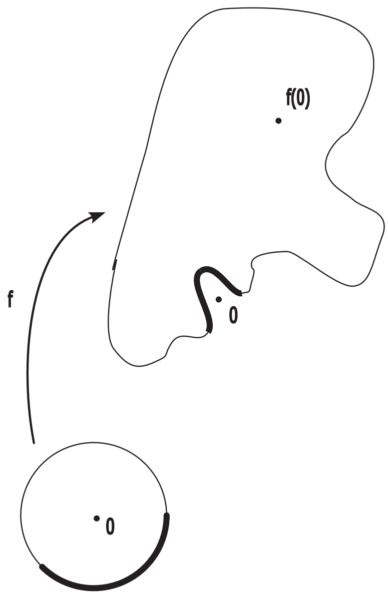
As $M(\Gamma_E) \leq KM(\Gamma'_E)$, we obtain

$$\sigma(E) \leq C(n,K)(\log M)^{1-n}.$$

Finally, for $1 < M \leq 3$, clearly $\sigma(E) \leq \sigma(S^{n-1})(\log 3)^{n-1} (\log M)^{1-n}$.

Let then $x \in B^n$ be general. The desired estimate follows by mapping x to 0 by the Möbius transformation T_x, and applying the estimate from the first part of the proof to $g = f \circ T_x^{-1}$. □

The inequality (2) implies that it is not possible for a set of boundary points that go much closer to the origin than $f(0)$ to be of large measure. Namely, Lemma 1 says that a situation illustrated by the figure below can't occur.



Corrolary 1 (Astala-Koskela). *If f is quasiconformal in B^n, then*

$$|f(x)|^q \leq C\frac{1}{\sigma(S(x))}\int_{S(x)} |f(\omega)|^q d\sigma \qquad (3)$$

for all $x \in B^n$ and each $0 < q < \infty$. The constant C depends only on n, K, q.

Proof. Assume first that $f(x) \neq 0$ for all $x \in B^n$. We apply Lemma 1. If M_0 is so large that $C(n,K)(\log M_0)^{1-n} = 1/2$, the estimate in Lemma 1 gives

$$2\sigma(\{\omega \in S(x) : |f(\omega)| \geq |f(x)|/M_0\}) \geq \sigma(S(x)).$$

Consequently,

$$\int_{S(x)} |f(\omega)|^q d\sigma \geq |f(x)|^q M_0^{-q} \sigma$$

$$\times (\{\omega \in S(x) : |f(\omega)| \geq |f(x)|/M_0\})$$

$$= C(n,K)|f(x)|^q \sigma(S(x)).$$

If $f(x) = 0$ for some $x \in B^n$, we may choose a point y in the complement of $f(B^n)$ so that $|y| \leq |f(\omega)|$ for all $\omega \in S^{n-1}$. Applying the above estimate to $f - y$ we get

$$C_q\, \sigma(S(x))|\,|f(x)|^q - |y|^q| \leqslant \sigma(S(x))|f(x) - y|^q$$

$$\leqslant C \int_{S(x)} |f(\omega) - y|^q d\sigma$$

$$\leqslant C \left[\int_{S(x)} |f(\omega)|^q d\sigma + \int_{S(x)} |y|^q d\sigma \right].$$

Now we have

$$C_q\, \sigma(S(x))|f(x)|^q$$

$$\leqslant C \int_{S(x)} |f(\omega)|^q d\sigma + (C + C_q) \int_{S(x)} |y|^q d\sigma$$

$$\leqslant C \int_{S(x)} |f(\omega)|^q d\sigma.$$

Zinsmeister's theorem follows now immediately: First of all notice that as $|f(r\omega)|, |f(\omega)| \leq |f^*(\omega)|$, the condition 3. of Theorem 3 implies conditions 1. and 2. An easy exercise to the reader is to use Fatou lemma and prove that the condition 2. yields the condition 1. It remains to prove that the condition 1. of Theorem 3 implies the condition 3.

So, assume the condition 1. By taking supremum when $x \in \Gamma(\omega)$ in inequality (3) we obtain

$$f^*(\omega)^q \leqslant C \cdot \sup_{x \in \Gamma(\omega)} \left(\frac{1}{\sigma(S(x))} \int_{S(x)} |f(\omega)|^q d\sigma \right)$$

$$\leqslant C\, M(|f|^q(\omega))$$

where M is Hardy-Littlewood maximal function on the sphere S^{n-1} :

$$M(f)(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)|\, dy.$$

Since M is a bounded operator on $L^s(S^{n-1})$, for all $s = \frac{p}{q} > 1$ we obtain for $q < p$,

$$\int_{S^{n-1}} f^*(\omega)^p d\sigma = \int_{S^{n-1}} (f^*(\omega)^q)^{p/q} d\sigma$$

$$\leqslant C \int_{S^{n-1}} (M(|f|^q(\omega)))^{p/q} d\sigma$$

$$\leqslant C \int_{S^{n-1}} (|f(\omega)|^q)^{p/q} d\sigma$$

$$= C \int_{S^{n-1}} |f(\omega)|^p d\sigma = \|f\|_{L^p}^p.$$

This finishes the proof that $f(\omega) \in L^p(S^{n-1})$ implies $f^*(\omega) \in L^p(S^{n-1})$, i.e., that the condition 1. of Theorem 3 implies the condition 3.

5. Further generalizations

One of the possible extensions of Theorem 3 is in the case of quasiregular maps with bounded multiplicity. More precisely, we would like to pose the following problem whose solution would require a deferent approach from that taken in [3].

Problem 1. *Does Theorem 3 hold when quasiconformality is replaced with quasiregularity with bounded multiplicity?*

It turns out, however, that the proof from [3] for the case of proper quasiregular maps can be fairly easily adjusted.

Recall that proper mappings are continuous mappings with the property that inverse of a compact set is compact. For discrete and open maps condition that f is proper is equivalent to the condition that f is closed, as well as to the condition that f is boundary preserving [11].

For example, proper analytic mappings of the unit disk onto the unit disk are precisely finite Blaschke products, i.e. mappings of the form

$$\prod_n \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \bar{a}_n z}.$$

To modify the crucial steps of the proof from [3], first note that for a proper quasiregular mapping $f : \overline{B}^n \longrightarrow \mathbb{R}^n$ we have that

$$f(S^{n-1}) = \partial f(\overline{B}^n) \tag{4}$$

and that proper quasiregular mappings are necessarily of bounded multiplicity.

For K-quasiregular map $f : \Omega \longrightarrow \Omega'$ of multiplicity bounded by N, we have the modulus estimate similar to that for K-quasiconformal maps, namely $M(\Gamma_E) \leqslant K \cdot N \cdot M(\Gamma'_E)$ for all rings E such that $\overline{E} \subset \Omega$.

Thus, the constants will now depend on dimension n, multiplicity N and the quasiconformality constant K, but otherwise the proof goes mostly unchanged.

However, there is one more key point where the condition $f(S^{n-1}) = \partial f(\overline{B}^n)$ is essential.

Namely, when proving Corollary 1 case $f(x) = 0$ for $x \in B^n$ and f is only N-quasiregular we may not be able to choose a point y in the complement of $f(B^n)$ such that $|y| \leqslant |f(\omega|$ for all $\omega \in S^{n-1}$. But the condition (4) guaranties that $f(S^{n-1})$ is separated from 0 since the image of an internal point cannot be on a boundary of image.

The condition (4) will allow us to proceed like in the case of quasiconformal mappings:

$$\min_{y \in \partial f(B^n)} |y| = \inf_{\omega \in S^{n-1}} |f(\omega)|.$$

In [1] the program of extending Zinsmeister characterization theorem has been completed for a special class of quasiregular mappings in the plane. Every quasiregular mapping in the plane can be represented as $f = g \circ \phi$, where ϕ is quasiconformal and g is analytic.

Jerison and Weitsman have given in [6] an example of an analytic function $g \in \mathcal{H}^2$ and quasiconformal $\phi : \mathbb{D} \longrightarrow \mathbb{D}$ such that $f = g \circ \phi \notin \mathcal{H}^p_{qr}$ for any $p > 0$.

In [1] Adamowicz and González have considered the composition operator $C_\phi g = g \circ \phi$ for $g \in \mathcal{H}^p$ and found necessary and sufficient condition that guarantees that C_ϕ sends $\mathcal{H}^p$ to $\mathcal{H}^p_{qc}$ with bounded norm of C_ϕ for $0 < p < \infty$.

Namely, when $\phi^{-1}|_{\mathbb{T}}$ is a Lipschitz function then C_ϕ is a bounded operator and the converse is also true, as they show in the following theorem:

Theorem 4 (Adamowicz-González). *Let $\phi : \mathbb{D} \longrightarrow \mathbb{D}$ be a quasiconformal mapping and $0 < p < \infty$. Then $C_\phi : \mathcal{H}^p \longrightarrow \mathcal{H}^p_{qr}$ is a bounded operator if and only if $\phi^{-1}|_{\mathbb{T}}$ is a Lipschitz function.*

The paper [1] has also introduced the following class of quasiregular mappings:

$$\mathcal{F}_p := \{f : \mathbb{D} :\longrightarrow \mathbb{R}^2;\ f = g \circ f \text{ for some } g \in \mathcal{H}^p$$
$$\text{and } \phi^{-1}|_{\mathbb{T}} \text{ is a Lipschitz function}\}.$$

While their result shows that $\mathcal{F}_p \subset \mathcal{H}^p_{qr}$, there are mappings in $\mathcal{H}^p_{qr}$ that are not in $\mathcal{F}_p$. For instance if g is bounded regardless of what quasiconformal $\phi : \mathbb{D} \longrightarrow \mathbb{D}$ is $g \circ \phi$ is in $\mathcal{H}^p_{qc}$. In [1], they give the following more interesting example:

Theorem 5. *There exists a function $f = g \circ \phi \in \mathcal{H}^1_{qr}$ such that $\phi^{-1}|_{\mathbb{T}}$ is a Lipschitz but $g \notin \mathcal{H}^1$. Moreover, such a decomposition is unique in the sense that there are no $\tilde{g} \in \mathcal{H}^1$, $\tilde{g} \neq g$ and quasiconformal $\tilde{\phi}$, $\tilde{\phi} \neq \phi$ with Lipschitz $\tilde{\phi}^{-1}|_{\mathbb{T}}$, such that $f = \tilde{g} \circ \tilde{\phi}$.*

In [1] Adamowicz and González have shown that for K-quasiregular mappings from $\mathcal{F}_p$ we have the following analogue of Zinsmeister's characterization theorem:

Theorem 6 (Adamowicz-González). *Let $0 < p < \infty$ and f be a K-quasiregular mapping in $\mathcal{F}_p$. Then we have the following:*

1. *$f \in \mathcal{H}^p_{qr}$.*
2. *The non-tangential boundary values $f(\xi)$ exists for a.e. $\xi \in \mathbb{T}$ and $f(\xi) \in L^p(\mathbb{T})$.*
3. *The non-tangential maximal function $f^* \in L_p * (\mathbb{T})$.*
4. *If $2 \leqslant p \leqslant \frac{2K}{K-1}$, then it holds that $\int_{\mathbb{D}} |Df(z)|^p (1-|z|)^{p-1}\, dm < \infty$.*

It should be noted that Part 4 does not always hold when $0 < p < 2$ (see, for example, [4, Theorem 1]).

Zinsmeister's theorem has also been extended to quasiconformal maps in the Heinsenberg group $\mathbb{H}^1$. In the paper [2], the authors consider maps from the unit ball with the Korányi metric. Recall that the Korányi norm is given by

$$\|(z,t)\| = (|z|^4 + t^2)^{1/4}, \quad (z,t) \in \mathbb{H}^1$$

where Heinserberg group in coordinates (z,t), $z \in \mathbb{C}$, $t \in \mathbb{R}$ with the group product represented as

$$(z_1,t_1) \cdot (z_2,t_2) = (z_1 + z_2, t_1 + t_2 + 2Im(z_1\bar{z}_2)).$$

The Korányi distance is given by

$$d(p,q) := \|q^{-1} \cdot p\|, \quad p,q \in \mathbb{H}^1.$$

The paper [2] introduces the Hardy class $\mathcal{H}^p$ of maps of unit ball $B \subset \mathbb{H}^1$ in Korányi metric and shows the following.

Theorem 7 (Adamowicz-Fässler). *For every $K \geqslant 1$, there exists a constant $p_0 = p_0(K) > 0$ such that every K-qc map $f : B \longrightarrow f(B) \in \mathbb{H}^1$, belongs to $\mathcal{H}^p$ for all $0 < p < p_0$.*

For the class $\mathcal{H}^p$ which Adamowicz and Fässler introduce in [2], they prove the following analogue of the Zinsmeister and the Astala-Koskela theorem.

Theorem 8 (Adamowicz-Fässler). *Let $0 < p < \infty$. The following conditions are equivalent for a quasiconformal map $f : B \longrightarrow f(B) \in \mathbb{H}^1$:*

1. *$f \in \mathcal{H}^p$.*
2. *The non-tangential maximal function Mf of f belongs to $L^p(\mathcal{S}^3|_{\partial B})$.*
3. *The Korányi norm of the radial limit f^* of f belongs to $L^p(\mathcal{S}^3|_{\partial B})$.*

They apply Theorem 8 to characterize Carleson measures on B in terms of radial limits of quasiconformal maps on B. More precisely, in [2], Adamowicz and Fässler prove the following theorem.

Theorem 9 (Adamowicz-Fässler). *Suppose that μ is a Carleson measure on B and that $f : B \longrightarrow f(B) \subset \mathbb{H}^1$ is a K-quasiconformal mapping. Then*

$$\int_B \|f(q)\|^p \, d\mu(q) \leqslant C \int_{\partial B} \|f^*(\omega)\|^p \, d\mathcal{S}^3(\omega),$$

$$\text{for all } 0 < p < \infty, \tag{5}$$

where C depends only on p, K and the Carleson measure constant of μ. Conversely, for every $K \geqslant 1$ there exists $p(K) < 3$ such that if $p > p(K)$ is fixed and μ is a Borel measure for which (5) holds for all K-qc mappings, then μ is a Carleson measure.

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Details of Workshop/Conferences in India

For details regarding Mathematics Training and Talent Search Programme
Visit: `https://mtts.org.in/programme/mtts2021/`

For details regarding Annual Foundation Schools, Advanced Instructional Schools, NCM Workshops, Instructional Schools for Teachers, Teacher's Enrichment Workshops
Visit: `https://www.atmschools.org/`

Name: New Directions in Rational Points.
Date: January 7, 2024–January 12, 2024
Venue: BIRS-Chennai CMI, India.
Visit: http://www.numbertheory.org/ntw/N3.html

Name: 10th Annual International Conference on Algorithms and Discrete Applied Mathematics.
Date: February 15, 2024–February 17, 2024
Venue: IIT Bhilai, India.
Visit: https://events.iitbhilai.ac.in/caldam2024/

Name: 2nd International Conference on Nonlinear Dynamics and Applications.
Date: February 21, 2024–February 23, 2024
Venue: Sikkim Manipal Institute of Technology
Visit: https://icnda.in/

Name: International Conference on "Latest Advances in Computational and Applied Mathematics-2024 (LACAM-24)".
Date: February 21, 2024–February 24, 2024
Venue: IISER Thiruvananthapuram, Kerala
Visit: https://conference.iisertvm.ac.in/lacam-24/

Name: 26th Annual Conference of The Society of Statistics, Computer and Applications (SSCA) International Conference on Emerging Trends of Statistical Sciences in AI and its Applications (ETSSAA-2024).
Date: February 26, 2024– February 28, 2024
Venue: Department of Mathematics and Statistics & Centre for Artificial Intelligence Banasthali Vidyapith, Banasthali-304022, Rajasthan.
Visit: https://tinyurl.com/SSCA26ConfRegistrationhttps://drive.google.com/file/d/1O9CRTs3OP4N2JM38Hufh2MEw3Vu-yEBh/view?usp=sharing

Name: International Conference on Computations and Data Science.
Date: March 08, 2024–March 10, 2024
Venue: Department of Mathematics, IIT Roorkee.
Visit: https://www.iitr.ac.in/cods24/index.html?

Details of Workshop/Conferences Abroad

Name: Connections Workshop: Noncommutative Algebraic Geometry
Date: February 1, 2024–February 2, 2024
Venue: SLMath 17 Gauss Way, Berkeley, CA 94720, USA
Visit: www.msri.org/workshops/1054

Name: Introductory Workshop: Noncommutative Algebraic Geometry
Date: February 5, 2024–February 9, 2024
Venue: SLMath 17 Gauss Way, Berkeley, CA 94720, USA
Visit: www.msri.org/workshops/1055

Name: Mathematical Approaches For Connectome Analysis
Date: February 12, 2024–February 16, 2024
Venue: Institute For Pure And Applied Mathematics (IPAM), Los Angeles, CA, USA
Visit: www.ipam.ucla.edu/programs/workshops/mathematical-approaches-for-connectome-analysis/

Name: SIAM Conference On Uncertainty Quantification (UQ24)
Date: February 27, 2024–March 1, 2024
Venue: To Be Determined, Trieste, Italy.
Visit: www.siam.org/conferences/cm/conference/uq24

Name: Asymptotics In Complex Geometry: A Conference In Memory Of Steve Zelditch
Date: March 7, 2024–March 10, 2024
Venue: Northwestern University, Evanston, IL, USA
Visit: sites.google.com/view/asymptotics/

Name: Hot Topics: "Artin Groups And Arrangements: Topology, Geometry, And Combinatorics"
Date: March 11, 2024–March 15, 2024
Venue: SLMath 17 Gauss Way, Berkeley, CA 94720, USA
Visit: www.msri.org/workshops/1047

Name: Geometry, Statistical Mechanics, And Integrability
Date: March 11, 2024–June 14, 2024
Venue: Institute For Pure And Applied Mathematics (IPAM), Los Angeles, CA, USA
Visit: www.ipam.ucla.edu/programs/long-programs/geometry-statistical-mechanics-and-integrability/

Name: Spring School On Soliton Dynamics
Date: March 14, 2024–March 16, 2024
Venue: Texas A&M University, College Station, TX, USA.
Visit: sites.google.com/tamu.edu/solitons-spring-school

Name: 13th Ohio River Analysis Meeting (ORAM 13)
Date: March 16, 2024–March 17, 2024
Venue: University Of Kentucky, Lexington KY, USA
Visit: sites.google.com/view/oram-13/home

Name: AIM Workshop: Degree D Points On Algebraic Surfaces
Date: March 18, 2024–March 22, 2024
Venue: American Institute Of Mathematics, Pasadena, California, USA
Visit: aimath.org/workshops/upcoming/degreedsurface/

Name: Analysis On Fractals And Networks, And Applications
Date: March 18, 2024–March 22, 2024
Venue: CIRM, 163 Avenue De Luminy, Case 916 13288 Marseille Cedex 9, FRANCE.
Visit: conferences.cirm-math.fr/2950.html

Name: Multi-Scale Methods For Reactive Flow And Transport In Complex Elastic Media, Conference In Memory Of Prof. Andro Mikelic
Date: March 19, 2024–March 22, 2024
Venue: CAAC , Center For Advanced Academic Studies, Dubrovnik, Croatia.
Visit: web.math.pmf.unizg.hr/andromikelic/

Name: Workshop I: Statistical Mechanics And Discrete Geometry
Date: March 25, 2024–March 29, 2024
Venue: Institute For Pure And Applied Mathematics (IPAM), Los Angeles, CA, USA
Visit: www.ipam.ucla.edu/programs/workshops/workshop-i-statistical-mechanics-and-discrete-geometry/

Name: Modern Aspects Of Harmonic Analysis On Lie Groups
Date: April 2, 2024–April 5, 2024
Venue: Georg-August-University GÖTtingen, GÖTtingen, Lower-Saxony/Germany.
Visit: jaeh.cc/SS2024/index.htm

Name: Recent Developments In Noncommutative Algebraic Geometry
Date: April 8, 2024–April 12, 2024
Venue: SLMath 17 Gauss Way, Berkeley, CA 94720, USA
Visit: www.msri.org/workshops/1075

Name: Workshop II: Integrability And Algebraic Combinatorics
Date: April 15, 2024–April 19, 2024
Venue: Institute For Pure And Applied Mathematics (IPAM), Los Angeles, CA, USA
Visit: www.ipam.ucla.edu/programs/workshops/workshop-ii-integrability-and-algebraic-combinatorics/

Name: Recent Developments In Commutative Algebra
Date: April 15, 2024–April 19, 2024
Venue: SLMath 17 Gauss Way, Berkeley, CA 94720, USA
Visit: www.msri.org/workshops/1060

Name: AIM Workshop: Higher-Dimensional Contact Topology
Date: April 15, 2024–April 19, 2024
Venue: American Institute Of Mathematics, Pasadena, California, USA
Visit: aimath.org/workshops/upcoming/highdimcontacttop/

Name: CRM Thematic Semester On "Geometric Analysis"
Date: April 15, 2024–April 29, 2024
Venue: Centre De RecherchesMathématiques, Université De Montréal, Québec, Canada.
Visit: www.crmath.ca/en/activities/{#}/type/activity/id/3880

Name: SIAM Conference On Data Mining (SDM24)
Date: April 18, 2024–April 20, 2024
Venue: Westin Houston, Memorial City, Houston, Texas, USA
Visit: www.siam.org/conferences/cm/conference/sdm24

Name: International Summit On Materials Science
Date: April 19, 2024–April 20, 2024
Venue: Tokyo, Japan.
Visit: materialsscience.averconferences.com/

Name: AIM Workshop: Post-Quantum Group-Based Cryptography
Date: April 29, 2024–May 3, 2024
Venue: American Institute Of Mathematics, Pasadena, California, USA
Visit: aimath.org/workshops/upcoming/postquantgroup/

Name: Advances In Lie Theory, Representation Theory And Combinatorics: Inspired By The Work Of Georgia M. Benkart
Date: May 1, 2024–May 3, 2024
Venue: SL Math 17 Gauss Way, Berkeley, CA 94720, USA
Visit: www.msri.org/workshops/1065/

Name: Workshop III: Statistical Mechanics Beyond 2D
Date: May 6, 2024–May 10, 2024
Venue: Institute For Pure And Applied Mathematics (IPAM), Los Angeles, CA, USA
Visit: www.ipam.ucla.edu/programs/workshops/workshop-iii-statistical-mechanics-beyond-2d/

Name: AIM Workshop: High-Dimensional Phenomena In Discrete Analysis
Date: May 13, 2024–May 17, 2024
Venue: American Institute Of Mathematics, Pasadena, California, USA
Visit: aimath.org/workshops/upcoming/highdimdiscrete/

Name: SIAM Conference On Applied Linear Algebra (LA24)
Date: May 13, 2024–May 17, 2024
Venue: Sorbonne Universite, Paris, France.
Visit: www.siam.org/conferences/cm/conference/la24

Name: SIAM Conference On Mathematical Aspects Of Material Science (MS24)
Date: May 19, 2024–May 23, 2024
Venue: Sheraton Pittsburgh Station Square, Pittsburgh, Pennsylvania, USA
Visit: www.siam.org/conferences/cm/conference/ms24

Name: Workshop IV: Vertex Models: Algebraic And Probabilistic Aspects Of Universality
Date: May 20, 2024–May 24, 2024
Venue: Institute For Pure And Applied Mathematics (IPAM), Los Angeles, CA, USA
Visit: www.ipam.ucla.edu/programs/workshops/workshop-iv-vertex-models-algebraic-and-probabilistic-aspects-of-universality/

Name: XXII GEOMETRICAL SEMINAR
Date: May 26, 2024–May 31, 2024
Venue: Vrnjaéka Banja, Serbia.
Visit: tesla.pmf.ni.ac.rs/people/geometrijskiseminarxxii/

Name: Representation Theory And Related Geometry: Progress And Prospects (On The Occasion Of Daniel K. Nakano's 60th Birthday)
Date: May 27, 2024–May 31, 2024
Venue: University Of Georgia, Athens, GA, USA.
Visit: sites.google.com/view/representation-theory-geometry

Name: SIAM Conference On Imaging Science (IS24)
Date: May 28, 2024–May 31, 2024
Venue: Westin Peachtree Plaza, Atlanta, Georgia, USA
Visit: www.siam.org/conferences/cm/conference/is24

Name: Computational Aspects Of Thin Groups
Date: June 3, 2024–June 14, 2024
Venue: IMS, National University Of Singapore.
Visit: ims.nus.edu.sg/events/computational-aspects-of-thin-groups/

Name: Séminaire De MathématiquesSupérieures 2024: "Flows And Variational Methods InRiemannian And Complex Geometry: Classical And Modern Methods (Montréal, Canada)"
Date: June 3, 2024–June 14, 2024
Venue: Montréal, Canada.
Visit: www.slmath.org/summer-schools/1061

Name: BIOMATH 2024: International Conference On Mathematical Methods And Models In Biosciences
Date: June 16, 2024–June 22, 2024
Venue: Cutty Sark Resort, Scottburgh, South Africa.
Visit: biomath.bg/2024

Name: Open Communications In Nonlinear Mathematical Physics - 2024
Date: June 23, 2024–June 29, 2024
Venue: Häcker's Grand Hotel, Bad Ems, Rhineland-Palatinate, Germany.
Visit: euler-ocnmp.de/

Name: New Perspectives In Computational Group Theory
Date: June 24, 2024–June 26, 2024
Venue: University Of Warwick.
Visit: sites.google.com/view/newperspectivescgt/home

Name: SIAM Conference On Nonlinear Waves And Coherent Structures (NWCS24)
Date: June 24, 2024–June 27, 2024
Venue: Lord Baltimore Hotel, Baltimore, MD, USA
Visit: www.siam.org/conferences/cm/conference/nwcs24

Name: ICERM Workshop: Queer In Computational And Applied Mathematics
Date: June 24, 2024–June 28, 2024
Venue: ICERM (Providence, Rhode Island), USA
Visit: icerm.brown.edu/topical_workshops/tw-24-qcam/

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